Some Correlation Inequalities for Ising Antiferromagnets

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Abstract. We prove some inequalities for two-point correlations of Ising antiferromagnets and derive inequalities relating correlations of ferromagnets to correlations of antiferromagnets whose interactions and field strengths have equal magnitudes. The proofs are based on the method of duplicate spin variables introduced by J. Percus and used by several authors (Refs. 3-8) to derive correlation inequalities for Ising ferromagnets.

1. Introduction

Correlation inequalities have played an important role in statistical mechanics, especially as applied to ferromagnetic Hamiltonians. It is the purpose of this note to apply known techniques to obtain some correlation inequalities for antiferromagnets.

Let $H_1(\sigma)$ be a ferromagnetic Hamiltonian for finite volume Λ in $\mathbf{Z}^d$ given by

$$H_1(\sigma) = \sum_{(i,j)\subset\Lambda} J_{ij}\sigma_i\sigma_j + \sum_{\substack{i\in\Lambda \\ j\notin\Lambda}} J_{ij}\sigma_i\bar{\sigma}_j - h\sum_{i\in\Lambda}\sigma_i \qquad (1.1)$$

and $H(x)$ a corresponding antiferromagnetic Hamiltonian for Λ given by

$$H(x) = \sum_{(i,j)\subset\Lambda} K_{ij}x_i x_j + \sum_{\substack{i\in\Lambda \\ j\notin\Lambda}} K_{ij}x_i\bar{x}_j - h\sum_{i\in\Lambda}x_i \qquad (1.2)$$

where the first sums in (1.1) and (1.2) are over all distinct pairs (i,j) in Λ, $\{\bar{x}_j\}$ and $\{\bar{\sigma}_j\}$ represent boundary configurations, and $K_{ij} = (-1)^{|i|+|j|} J_{ij}$. Here $|i| = |(i_1, i_2,\ldots,i_d)| = |i_1|+\cdots+|i_d|$ and $x_i, \sigma_i = \pm 1$, and $J_{ij} \le 0$. We will consider (1.2) with the change of variable $x_i = (-1)^{|i|} s_i$ and denote the resulting Hamiltonian by $H_2(s)$ so that

$$H_2(s) = \sum_{(i,j)\ \Lambda} J_{ij} s_i s_j + \sum_{\substack{i\ \Lambda \\ j\ \Lambda}} J_{ij} s_i \bar{s}_j - \sum_{i\ \Lambda} k_i s_i \qquad (1.3)$$

where $k_i = (-1)^{|i|} h$. We will denote expectations with respect to the finite volume Gibbs states corresponding to (1.1) and (1.3) by $\langle \cdot \rangle_F$ and $\langle \cdot \rangle_A$ respectively; boundary configurations will always be assumed fixed.

In Section 2 of this paper we derive Lebowitz-type inequalities[3] which allow the comparison of correlations corresponding to Hamiltonians (1.1) and (1.3). When h=0 (1.1) and (1.3) are equal and have equal correlation functions. When $h \neq 0$ $H_1(\sigma)$ has a unique phase for all temperatures, which implies decay properties of truncated correlation functions for $H_1(\sigma)$. Our inequalities are valid for all h, even though $h \neq 0$ includes both single and multiple phase regions for H(x) (see, for example, Ref. 1).

In Section 3 we prove some monotonicity properties for two-point correlations corresponding to (1.3). The method of proof is based on the techniques used by Messager and Miracle-Sole[4] to derive, among other things, monotonicity properties for correlations corresponding to nearest neighbor ferromagnetic interactions. We make some modifications of their methods to accommodate nonnearest neighbor interactions and nonpositive external fields $\{k_i\}$. We allow our Hamiltonians to have infinite range, but our inequalities are weaker than those of Ref. 4 for the ferromagnetic case. We note that Hegerfeldt[2] generalized some of the monotonicity results in Ref. 4 for ferromagnetic correlations, but the methods of Ref. 2 do not seem to extend readily to antiferromagnetic interactions.

2. Comparison of Correlations

Let two Ising spin Hamiltonians $H_a(\sigma)$ and $H_b(s)$ for volume Λ in $\mathbf{Z}^d$ be given by

$$H_a(\sigma) = \sum_{(i,j) \in \Lambda} J_{ij}\sigma_i\sigma_j - \sum_{i \in \Lambda} h_i\sigma_i \qquad (2.1)$$

$$H_b(s) = \sum_{(i,j) \in \Lambda} J_{ij} s_i s_j - \sum_{i \in \Lambda} k_i s_i \qquad (2.2)$$

where $J_{ij} \leq 0$ and σ_i, $s_i = \pm 1$ for all $i, j \in \mathbf{Z}^d$. The external field variables h_i and k_i are of the form

$$h_i = h'_i - \sum_{j \notin \Lambda} J_{ij}\bar{\sigma}_j \qquad (2.3)$$

$$k_i = k'_i - \sum_{j \notin \Lambda} J_{ij}\bar{s}_j \qquad (2.4)$$

where $\{ \bar{\sigma}_j \}$ and $\{ \bar{s}_j \}$ may be interpreted as fixed boundary configurations. Correlation functions, for a finite set B in $\mathbf{Z}^d$, with respect to the finite volume Gibbs measures for (2.1) and (2.2) will be denoted by $\prod_{i \in B} \sigma_i \equiv \sigma_B$ and $\prod_{i \in B} s_i \equiv s_B$ respectively.

Let

$$H_i = h_i + k_i \quad \text{and} \quad K_i = h_i - k_i \,. \qquad (2.5)$$

Define spin variables q_i and t_i taking values -1,0,+1 by

$$t_i = 1/2\,(\sigma_i + s_i) \quad \text{and} \quad q_i = 1/2\,(\sigma_i - s_i). \qquad (2.6)$$

Let $\langle \cdot \rangle$ denote expectations with respect to the product measure

$$\mu(\sigma,s) = \frac{1}{Z_a(\Lambda)}\frac{1}{Z_b(\Lambda)} \exp\{-\beta[H_a(\sigma) + H_b(s)]\} \qquad (2.7)$$

where $Z_a(\Lambda)$ and $Z_b(\Lambda)$ are the partition functions for $H_a(\sigma)$ and $H_b(\Lambda)$ respectively. For finite sets A, B in $\mathbf{Z}^d$, let $t_A = \prod_{i \quad A} t_i$ and $s_B = \prod_{i \quad B} q_i$. The following theorem, though not stated in this generality, was proved by Lebowitz in Ref. 3 (see also Percus[5] and Sylvester[7]).

Theorem 2.1 If H_i , $K_i \geq 0$ for all i Λ, then for any two subsets A, B in Λ,

a) t_A , q_A ≥ 0

b) $t_A t_B \geq t_A \quad t_B$

c) $q_A q_B \geq q_A \quad q_B$

d) $q_A \quad t_B \geq q_A t_B$.

Remark 2.1 By symmetry, it may be assumed that H_i , $K_i \leq 0$, in which case inequalities a) - d) are modified by replacing each q_i by $-q_i$ and each t_i by $-t_i$.

Corollary 2.1 With the same assumptions as in Theorem 2.1,

a) t_A decreases and q_A increases as each K_i increases
b) t_A increases and q_A decreases as each H_i increases.

proof . This follows by differentiating t_A and q_A by H_i or K_i and applying b), c), or d) of Theorem 2.1.

A substantial generalization of part a) of the following Corollary was proved by Lebowitz in Ref. 8 (see also Griffiths[9]).

Corollary 2.2 If H_i , $K_i \geq 0$ for all i Λ, then for any subset B in Λ, and any i, j Λ,

a) $\sigma_B \geq | \quad s_B \quad |$

b) $\sigma_i \sigma_j - \sigma_i \quad \sigma_j \leq s_i s_j - s_i \quad s_i$

proof. The following identities, where σ_i and s_i may be complex numbers are well known and easily verified (see for example Ref. 2):

$$\prod_{i \in A} \sigma_i + \prod_{i \in A} s_i = 2^{-|A|+1} \sum_{\substack{B \subseteq A \\ |B| \text{even}}} \prod_{i \in B} (\sigma_i - s_i) \prod_{i \in A \setminus B} (\sigma_i + s_i) \qquad (2.8)$$

$$\prod_{i \in A} \sigma_i - \prod_{i \in A} s_i = 2^{-|A|+1} \sum_{\substack{B \subseteq A \\ |B| \text{odd}}} \prod_{i \in B} (\sigma_i - s_i) \prod_{i \in A \setminus B} (\sigma_i + s_i) \qquad (2.9)$$

where |A| denotes the cardinality of A. Identifying σ_i and s_i as Ising spin variables yields,

$$\prod_{i \in A} \sigma_i - \prod_{i \in A} s_i = 2 \sum_{\substack{B \subseteq A \\ |B| \text{odd}}} q_B t_{A \setminus B} \qquad (2.10)$$

and

$$\prod_{i \in A} \sigma_i + \prod_{i \in A} s_i = 2 \sum_{\substack{B \subseteq A \\ |B| \text{even}}} q_B t_{A \setminus B} \qquad (2.11)$$

Taking · expectations of (2.10) and (2.11) yields part a) of the corollary. The proof of part b) follows directly from part d) of Theorem 2.1 with A = {i} and B = {j}. This completes the proof.

We now consider the special case for the Hamiltonians (2.1) and (2.2) where in equation (2.3), $h'_i \equiv h$ for all i and some constant h, and in equation (2.4) $k'_i = (-1)^{|i|} h$. With these identifications $H_a(\sigma)$ equals $H_1(\sigma)$, given by equation (1.1), and $H_b(s)$ equals $H_2(s)$, given by equation (1.3). The following corollary is now an immediate consequence of Corollary 2.2.

Corollary 2.3 Let $h \geq 0$. Assume that for all i Λ

1) $h \geq 1/2 \sum_j \; \Lambda \, J_{ij} \, (\bar{\sigma}_j + \bar{s}_j)$

2) $\sum_j \; \Lambda \, J_{ij} \, (\bar{\sigma}_j - \bar{s}_j) \leq 0$.

Then for any subset B of Λ, and any i,j Λ, the following inequalities for the correlations of the Hamiltonians given by (1.1) and (1.3) hold:

a) $\prod_i \; B \, \sigma_i \; F \geq | \; \prod_i \; B \, s_i \; A \, |$

b) $\sigma_i \sigma_j \; F - \; \sigma_i \; F \; \sigma_j \; F \leq \; s_i s_j \; A - \; s_i \; A \; s_j \; A$.

Remark 2.2 An analogous statement may be made for $h \leq 0$ (see Remark 2.1).

Remark 2.3 The hypotheses to Corollary 2.3 are satisfied, for example, if $\bar{\sigma}_j \equiv +1$ for all j Λ. In this case $\bar{s}_j$ for j Λ may be chosen arbitrarily. It is also easily shown that if $H_1(\sigma)$ and H_2 (s) both have empty or both have periodic boundary conditions, then a) and b) of Corollary 2.3 hold.

3. Monotonicity Properties for Antiferromagnets

In this section we prove some monotonicity properties for two point correlations for antiferromagnets. Denote by H(s) the Hamiltonian,

$$H(s) = \sum_{\Lambda} J_{ij} s_i s_j - \sum_i \; \Lambda \, k_i s_i \qquad (3.1)$$

where here and below $\sum_{\Lambda}$ means sum over all distinct pairs (i,j) in the subset Λ in $\mathbf{Z}^d$. We also assume that $J_{ij} \leq 0$ and that J_{ij} is a function of $\| i - j \|$, the Euclidean norm of i - j. The external field k_i is given by

$$k_i = k'_i - \sum_{j \ \Lambda} J_{ij} \bar{s}_j \qquad (3.2)$$

for some boundary configuration $\{ \bar{s}_j \}$, where $k'_i = (-1)^{|i|} h$ for some $h \geq 0$, so that H(s) is equal to the antiferromagnetic Hamiltonian (1.3). In this section, denote by or $_\Lambda$ expectations with respect to the finite volume Gibbs state determined by (3.1).

Theorem 3.1 Let ϕ: $\mathbf{Z}^d \rightarrow \mathbf{Z}^d$ by $\phi(i_1,\ldots, i_d) = (-(i_1+2), i_2,\ldots, i_d)$. Let Λ be a rectangle in $\mathbf{Z}^d$ invariant under ϕ and let the boundary configuration $\{ \bar{s}_j \}$ be invariant under ϕ. Suppose also that $|J_{ij}| \geq 1/2 \, | J_{i\phi(j)} |$. Then for any i,j Λ with $i_1, j_1 \geq 0$,

$$s_i \, s_j \ \geq \ s_i \, s_{\phi(j)} \qquad (3.3)$$

Remark 3.1 It is also possible to consider periodic boundary conditions. If 0 Λ (3.3) and the symmetry of the finite volume Gibbs State imply

$$\langle s_0 s_j \rangle \geq \langle s_0 s_{(j_1+2, j_2, \ \ , j_d)} \rangle , \qquad (3.4)$$

for $j_1 \geq 0$, where s_0 is the spin at the origin of $\mathbf{Z}^d$.

proof. Let

$$\Lambda_+ = \{ i \ \ \Lambda : i_1 > -1 \}$$

$$\Lambda_0 = \{ i \ \ \Lambda : i_1 = -1 \}$$

$$\Lambda_- = \{ i \ \ \Lambda : i_1 < -1 \}.$$

Then $\Lambda = \Lambda_+ \ \ \Lambda_0 \ \ \Lambda_-$ and $\phi(\Lambda_+) = \Lambda_-$, $\phi(\Lambda_-) = \Lambda_+$, and $\phi(\Lambda_0) = \Lambda_0$. Denote $\phi(i)$ by $i\sim$. With this notation we can write,

$$\sum_{\Lambda} J_{ij}\, s_i s_j = \sum_{\Lambda+} J_{ij}\,(s_i s_j + s_{i\sim} s_{j\sim}) + \sum_{i \quad \Lambda o} \sum_{j \quad \Lambda+} J_{ij}\,(s_i s_j + s_{i\sim} s_{j\sim})$$

$$+ 1/2 \sum_{\Lambda o} J_{ij}\,(s_i s_j + s_{i\sim} s_{j\sim}) + 1/2 \sum_{\Lambda+} J_{ij\sim}\,(s_i s_{j\sim} + s_{i\sim} s_j)$$

$$+ 1/2 \sum_{i \quad \Lambda+} J_{ii\sim}\,(s_i s_{i\sim} + s_i s_{i\sim}). \qquad (3.5)$$

The last two terms on the right side of (3.5) may be rewritten as,

$$1/2 \sum_{\Lambda+} J_{ij\sim}\,(s_i + s_{i\sim})(s_j + s_{j\sim}) - 1/2 \sum_{\Lambda+} J_{ij\sim}\,(s_i s_j + s_{i\sim} s_{j\sim})$$

$$+ 1/2 \sum_{i \quad \Lambda+} J_{ii\sim}\,(s_i + s_{i\sim})^2 - \sum_{i \quad \Lambda+} J_{ii\sim}\,.$$

Let

$$t_i = 1/2\,(s_i + s_{i\sim}) \text{ and } q_i = 1/2\,(s_i - s_{i\sim}) \qquad (3.6)$$

so that

$$s_i s_j + s_{i\sim} s_{j\sim} = 2(t_i t_j + q_i q_j). \qquad (3.7)$$

Combining (3.5) - (3.7) and observing that $q_i = 0$ if i Λ_0 gives,

$$\sum_{\Lambda} J_{ij}\, s_i s_j = \sum_{\Lambda+}(2J_{ij} - J_{ij\sim})\, q_i q_j + \sum_{\Lambda+} 2(J_{ij} + J_{ij\sim})\, t_i t_j$$

$$+ 2 \sum_{i \quad \Lambda o} \sum_{j \quad \Lambda+} J_{ij}\, t_i t_j + \sum_{\Lambda o} J_{ij}\, t_i t_j$$

$$+ 2 \sum_{\Lambda+} J_{ii\sim} t_i^2 - \sum_{\Lambda+} J_{ii\sim}\,. \qquad (3.8)$$

Now define $H_i = k_i + k_{i\sim}$ and $K_i = k_i - k_{i\sim}$ so that

$$k_i s_i + k_{i\sim} s_{i\sim} = H_i t_i + K_i q_i\,. \qquad (3.9)$$

From the definition of ϕ and k_i and the invariance of the boundary conditions $\{s_j\}$ under ϕ, it follows that $K_i = 0$ for all i Λ. Thus to within an additive constant,

$H(s) = H^1(q) + H^2(t)$, where

$$H^1(q) = \sum_{\Lambda+} N_{ij}\, q_i q_j$$

$$H^2(t) = \sum_{\Lambda+ \quad \Lambda o} M_{ij}\, t_i t_j + 2 \sum_{i \quad \Lambda+} J_{ii\sim}\, t_i^2 + \sum_{i \quad \Lambda+} H_i t_i + 1/2 \sum_{i \quad \Lambda o} H_i t_i \qquad (3.10)$$

and N_{ij} and M_{ij} are nonpositive.

From the definitions of q_i and t_i it follows that $t_i = 0$ iff $q_i = \pm 1$ and $q_i = 0$ iff $t_i = \pm 1$. Also if i Λ_0, then $t_i = \pm 1$. For any functions $\varphi(q)$ and $\psi(t)$,

$$\langle \varphi(q)\psi(t)\rangle = \frac{1}{Z_\Lambda(s)} \sum_{(q,t)} \varphi(q)\psi(t)\exp\{-\beta[H^1(q) + H^2(t)]\} \tag{3.11}$$

where the sum in (3.11) is over all pairs $q = \{q_i\}_i$ Λ_+ and $t = \{t_i\}_i$ Λ_+ Λ_o such that $t_i = \pm 1$ if i Λ_0, and $q_i = 0$ iff $t_i = \pm 1$ otherwise. Equation (3.11) may be rewritten as,

$$\langle \varphi(q)\psi(t)\rangle = \frac{1}{Z_\Lambda(s)} \sum_q \sum_t \sum_{\Lambda_0 \;\; A \;\; \Lambda_0 \;\; \Lambda_+} \varphi(q)\chi_A(q)\psi(t)\chi_{A^c}(t)\exp\{-\beta[H^1(q) + H^2(t)]\} \tag{3.12}$$

where the sums on q and t now include the values ±1 for q_i and t_i , but not zero, $A^c = (\Lambda_+ \;\; \Lambda_0) \setminus A$, and

$$\chi_A(q) = \begin{cases} 1, & \text{when } q_i = 0 \text{ iff } i \;\; A \\ 0, & \text{otherwise} \end{cases}.$$

For any A Λ_0 Λ_+, let

$$P(A) = \frac{\sum_t \chi_{A^c}(t)\exp[-\beta H^2(t)] \sum_q \chi_A(q)\exp[-\beta H^1(q)]}{Z_\Lambda(s)} \tag{3.13}$$

$$\equiv \frac{Z_A(t) Z_{A^c}(q)}{Z_\Lambda(s)}$$

where $Z_{A^c}(q)$ and $Z_A(t)$ are the usual Ising partition functions respectively for $H^1(q)$ and $H^2(t)$ with q_i , $t_i = \pm 1$. Then (3.12) may be rewritten as

$$\varphi(q)\psi(t) = \sum_{\Lambda_0 \;\; A \;\; \Lambda_0 \;\; \Lambda_+} P(A) \;\; \varphi(q)\chi_A(q) \;\; {}_{A^c,q} \;\; \psi(t)\chi_{A^c}(t) \;\; {}_{A,t} \tag{3.14}$$

where $\varphi(q)\chi_A(q)_{A^c,q} = Z_{A^c}(q)^{-1} \sum_q \chi_A(q)\, \varphi(q) \exp[-\beta H^1(q)]$ and $\psi(t)\chi_{A^c}(t)_{A,t}$ has an analogous expression. Let $\psi(t) \equiv 1$ and $\varphi(q) = q_i\, q_j$. Then

$$q_i q_j = \sum_A P(A)\ q_i q_j \chi_A(q)_{A^c,q} \geq 0 \qquad (3.15)$$

since by Griffith's inequality each term in the sum is nonnegative. Thus

$$s_i\, s_j + s_{i\sim}\, s_{j\sim} \geq s_{i\sim}\, s_j + s_i\, s_{j\sim} \qquad (3.16)$$

and the conclusion of the theorem now follows from the invariance of Λ and the boundary conditions under the reflection ϕ. This completes the proof.

The following corollary, establishing a version of the Percus inequality or first Lebowitz inequality, follows immediately from the arguments leading up to equation (3.14).

Corollary 3.1 With the hypotheses and notation of Theorem 3.1,

$$\prod_{i \in A} (s_i - s_{\varphi(i)}) \geq 0 \qquad (3.17)$$

for any A in Λ_+.

Theorem 3.2 Let Ψ: $\mathbf{Z}^d \to \mathbf{Z}^d$ by $\Psi(i_1,...,i_d) = (-(i_1+1), i_2,...,i_d)$. Let a rectangle Λ in $\mathbf{Z}^d$ and a boundary condition $\{\bar{s}_j\}_{j \in \Lambda^c}$ be invariant under Ψ. Then for any $i,j \in \Lambda$ with $i_1, j_1 \geq 0$

$$s_i\, s_j \geq -\ s_i\, s_{\Psi(j)} \ . \qquad (3.18)$$

The proof of Theorem 3.2 is similar to and simpler than the proof of Theorem 3.1. In this case $\Lambda_+ = \{i \in \Lambda: i \geq 0\}$, $\Lambda_- = \{i \in \Lambda: i \leq -1\}$, and Λ_0 is empty. With analogous notation as in the proof of Theorem 3.1, $H_i \equiv 0$ and it follows that $\prod_{i\ A} t_i \geq 0$ for any subset A of Λ_+. The case in which J_{ij} is nonzero only for $\| i - j \| = 1$ was essentially contained in the proof of an analogous theorem (Theorem 1) of Messager and Miracle-Sole[4] for Ising ferromagnets.

Corollary 3.2 If J_{ij} satisfies the conditions of Theorem 3.1, then

1) $$s_0 s_j {}^{\pm}_{\infty} \geq s_0 s_{(j_1+2, j_2, \ldots, j_d)} {}^{\pm}_{\infty}$$

2) $$s_0 s_j {}^{\pm}_{\infty} \geq - s_0 s_{(j_1+1, j_2, \ldots, j_d)} {}^{\pm}_{\infty}$$

for any j with $j_1 \geq 0$, where $s_0 s_j {}^{\pm}_{\infty} = \lim_{\Lambda \uparrow Z^d} s_0 s_j {}_{\Lambda}$ with boundary conditions $\bar{s}_j \equiv +1$ or $\bar{s}_j \equiv -1$ for all j Λ^c and the limit may be taken along any sequence Λ_n increasing to $\mathbf{Z}^d$.

proof. Let $\rho_j = 1/2\,(s_j + 1)$. Then $\rho_0 \rho_j$ is an increasing function in the sense used in the FKG inequalities. Since $4\rho_0\rho_j = [s_0 s_j + s_0 + s_j + 1]$ and s_0 and s_j are also increasing, it follows that lim $s_0\, s_j\ {}_{\Lambda}$ exists along any sequence Λ_n increasing to $\mathbf{Z}^d$. Let Λ_n be as in Theorem 3.1 and let $\tilde{\Lambda}_n$ be the reflection of Λ_n across the hyperplane $j_1 = 0$. Then

$$s_0 s_{\varphi(j)}\ {}_{\Lambda_n} = s_0 s_{(j_1+2, j_2, \ldots, j_d)}\ {}_{\tilde{\Lambda}_n}$$

Inequality 1) now follows by applying Theorem 3.1 and taking limits. The proof of 2) is similar.

Remark 3.2 We note that other axes and reflections may be used in Theorems 3.1 and 3.2; the crucial point is that H_i or K_i or both (as in the case of ferromagnetic interactions) must be nonnegative (see (3.9)).

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4. References

1. Dobrushin, R. L.: Phase diagram of the two-dimensional Ising antiferromagnet (computer-assisted proof). Commun. Math. Phys. 102, 89-103 (1985)

2. Hegerfeldt, G. H.: Correlation inequalities for Ising ferromagnets with symmetries. Commun. Math. Phys. 57, 259-266 (1977)

3. Lebowitz, J. L.: GHS and other inequalities. Commun. Math. Phys. 35, 87-92 (1974)

4. Messager, A. and Miracle-Sole, S.: Correlation functions and boundary conditions in the Ising ferromagnet. J. Stat. Phys. 17 no. 4, 245-262 (1977)

5. Percus, J.: Correlation Inequalities for Ising spin lattices. Commun. Math. Phys. 40, 283-308 (1975)

6. Schrader, R.: New correlation inequalities for the Ising model and P(φ) theories. Phys. Rev. B 15 2798 (1977)

7. Sylvester, G. S.: Inequalities for Continuous-Spin Ising Ferromagnets. J. Stat. Phys. 15, 327-341 (1976)

8. Lebowitz, J. L.: Number of Phases in One Component Ferromagnets. In: Lecture Notes in Physics 80, Springer 1977

9. Griffiths, R. B.: Phase Transitions. In: Statistical Mechanics and Quantum Field Theory. Les Houches Lectures 1970. Gordon and Breach Science Publishers, New York, London, Paris 1971