LUAN C. TA

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ANSWERS

22. To show that $f(x) = \dfrac{2x-1}{2x(1-x)}$ is a bijective function from $(0,1)$ to $\mathbb{R}$, we need to show:

(i) $f(x)$ is 1-1.

(ii) $f(x)$ is onto.

We first show that $f(x)$ is 1-1.

We have
$$f(x) = \frac{2x-1}{2x(1-x)}$$
$$= \frac{x-1+x}{2x(1-x)}$$
$$= \frac{1}{2(1-x)} - \frac{1}{2x}$$

Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$

Then $y_1 = y_2$ implies:
$$\frac{1}{2(1-x_1)} - \frac{1}{2x_1} = \frac{1}{2(1-x_2)} - \frac{1}{2x_2}$$
$$\Leftrightarrow \frac{1}{2(1-x_1)} - \frac{1}{2(1-x_2)} - \left(\frac{1}{2x_1} - \frac{1}{2x_2}\right) = 0$$
$$\Leftrightarrow \frac{1}{2} \cdot \frac{x_1 - x_2}{(1-x_1)(1-x_2)} + \frac{1}{2} \cdot \frac{x_1 - x_2}{x_1 x_2} = 0$$
$$\Leftrightarrow \frac{1}{2}(x_1 - x_2)\left[\frac{1}{(1-x_1)(1-x_2)} + \frac{1}{x_1 x_2}\right] = 0 \quad (*)$$

Since $x_1, x_2 \in (0,1) \Rightarrow 0 < x_1 < 1$ and $0 < x_2 < 1$

Thus $\dfrac{1}{(1-x_1)(1-x_2)} + \dfrac{1}{x_1 x_2} > 0$

So (*) implies $x_1 - x_2 = 0$

So $x_1 = x_2$, and this show that $f(x)$ is 1-1.

We now prove that $f(x)$ is also onto.

Let $y \in \mathbb{R}$ and $y = f(x) = \dfrac{2x-1}{2x(1-x)}$

So $2xy(1-x) = 2x-1$

$\Leftrightarrow \quad 2yx^2 + 2(1-y)x - 1 = 0 \quad (**)$

If $y=0$ then $(**)$ implies $x = \frac{1}{2}$

If $y \neq 0$ then the equation ~~(*)~~ $(**)$ has two roots,

that are: $x_1 = \dfrac{1}{2} - \dfrac{1+\sqrt{1+y^2}}{2y}$

$$x_2 = \frac{1}{2} - \frac{1-\sqrt{1+y^2}}{2y}$$

We will show that only $x_2 \in (0,1)$ and $x_1 \notin (0,1)$.

~~Since $y \neq 0$ then~~

Suppose that $x_1 \in (0,1)$, then:

$$0 < \frac{1}{2} - \frac{1+\sqrt{1+y^2}}{2y} < 1$$

$$\iff -\frac{1}{2} < -\frac{1+\sqrt{1+y^2}}{2y} < \frac{1}{2}$$

$$\iff \left| -\frac{1+\sqrt{1+y^2}}{2y} \right| < \frac{1}{2}$$

$$\iff \left| 1+\sqrt{1+y^2} \right| < |y|$$

Which is false because:

$$\left| 1+\sqrt{1+y^2} \right| > \sqrt{1+y^2} \geq |y|$$

So $x_1 \notin (0,1)$.

Now, suppose that $x_2 \in (0,1)$, then:

$$0 < \frac{1}{2} - \frac{1-\sqrt{1+y^2}}{2y} < 1$$

$$\iff -\frac{1}{2} < -\frac{1-\sqrt{1+y^2}}{2y} < \frac{1}{2}$$

$$\iff \left| -\frac{1-\sqrt{1+y^2}}{2y} \right| < \frac{1}{2}$$

$$\iff \left| 1-\sqrt{1+y^2} \right| < |y|$$

$$\Leftrightarrow \sqrt{1+y^2} - 1 < |y|$$

$$\Leftrightarrow \sqrt{1+y^2} < 1+|y|$$

$$\Leftrightarrow \left(\sqrt{1+y^2}\right)^2 < (1+|y|)^2$$

$$\Leftrightarrow 0 < 2|y|$$

$\frac{5}{5}$

which is true.

very good!

So $x_2 \in (0,1)$ And this shows that $f(x)$ is onto.

Thus $f(x)$ is ~~injective~~ bijective as required $\square$.