1) When deriving the Lorenz equations from the Navier-Stokes and Heat Diffusion equations, a standard procedure when dealing with an incompressible velocity ($\vec{\nabla} \cdot \vec{u} = 0$), is to introduce the stream function $\psi(x, z, t)$ such that
$$\vec{u} = \vec{\nabla} \times \vec{A}$$
with $\vec{A} = (0, \psi, 0) = \psi \hat{j}$.

a) Show that the velocity in the x-direction $u$ and z-direction $w$ are given by:
$$u = -\frac{\partial \psi}{\partial z}, \quad w = \frac{\partial \psi}{\partial x}$$

b) The equations encountered before the truncation implemented by Lorenz are quite complex. However Rayleigh found a simple solution for the steady state case and without non-linear terms. Consider these simple equations for $\psi$ and $\tau$ (the variation of the temperature from the linear height dependence):
$$-\frac{\partial \psi}{\partial x} = \nabla^2 \tau$$
$$0 = R \frac{\partial \tau}{\partial x} + \nabla^4 \psi$$
$$\nabla^4 \psi = \frac{\partial^4 \psi}{\partial x^4} + \frac{\partial^4 \psi}{\partial z^4} + 2 \frac{\partial^4 \psi}{\partial x^2 \partial z^2}$$

Find an expression for the Rayleigh number $R$ by direct substitution of the following solutions:
$$\psi = \psi_0 \sin \left( \frac{\pi ax}{h} \right) \sin \left( \frac{\pi z}{h} \right)$$
$$\tau = \tau_0 \cos \left( \frac{\pi ax}{h} \right) \sin \left( \frac{\pi z}{h} \right)$$

c) In the last page you will find Matlab commands which guide you step by step on how to go about creating a surface or contour plot for the function $\psi$.

Follow the steps and make sure that you get the displayed result.

Then calculate $u$ and $w$ and generate image and contour plots for each.

Finally produce one plot in which your superimpose a contour for $u$ on an image for $w$. Discuss if your results describe correctly the convection rolls. (to superpose one display on top of another use the "hold on" command).

2) Let $A$ be an $n \times n$ constant matrix. Show that the initial value problem, $\dot{X} = AX$ with $X(t_0) = X_0$ has the unique solution:
$$X(t) = e^{(t-t_0)A}X_0$$

You may use the fact that this result holds in the case that $t_0 = 0$ as explained in class.
Instructions to perform plots in Matlab for problem 1.

The purpose of this noes is to guide you through the steps on how to create a surface and a contour plot of the stream function \( \psi \), the x-velocity \( u \) and the z-velocity \( w \).

I am going to provide the steps that you can type one by one in the command line, but once you have checked that everything works the best is to create a script file which you can edit as you need.

If you do not understand a command or want to get more information about it please use

```
>> help command name
```

Wet set the amplitude \( \psi_o = 1 \) to simplify.

Set \( h=2 \) and \( a=h/\sqrt{2} \). You will be asked to play with this factor

```
>> h=2;
>> a=h/sqrt(h);
```

Set span of \( z \) (50 points from 0 to \( h \)) and \( x \) (200 points from 0 to 5\( h \)). Note that you can play with the extent of \( x \).

```
>> z=linspace(0, h, 50);
>> x=linspace(0, 5*h, 200);
```

Define a 50 by 200 matrix for the stream function

```
>> stream=zeros(50,200);
```

Now using a for loop input the values for the stream function

```
>> for i=1:50
    stream(i,:)=sin(pi*z(i)/h).*sin(pi*a*x/h);
end
```

Recall we are setting the amplitude to 1. Notice the array multiplication symbol (\( .* \)) so that you can obtain a 50 by 200 matrix from 50 entry vector multiplied by a 200 entry vector.

Display an image of the stream function

New figure

```
>> figure
```

Displays the image

```
>> imagesc(stream)
```

Displays the colorbar

```
>> colorbar
```
Sets the vertical axis with the origin at the bottom left corner

```matlab
>> set(gca, 'ydir', 'normal')
```

Notice that we have not bothered to put units on the axis.
To do a contour plot repeat the steps above but using the “contour” command

```matlab
>> figure
>> contour(stream)
>> colorbar
>> set(gca, 'ydir', 'normal')
```