

MATH 250: FINAL REVIEW, SPRING 2022

13.5) Lines and Planes in Space. 1-10, 11-26, 27-30, 31-37, 43-58, 61-64, 65-68, 71-72, 73-76, 77-80.

- (a) $\vec{r}(t) = (1 + 2t, -1 + 3t, 2 - t)$
(b) $\vec{r}(t) = \langle 2t, 2 - t, 3 - 2t \rangle$.
(c) $\vec{r}(t) = \langle -1 - 2t, 2 + t, t \rangle$.
- (a) $5(x - 2) - (y - 3) + 3(z - 1) = 0$ or $5x - y + 3z = 10$.
(b) $(x - 1) + (y - 2) - 2z = 0$ or $x + y - 2z = 3$.
(c) $x - y + z = 2$.

13.6) Cylinders and Quadric Surfaces. 1-6, 7-12, 15-20, 21-28, 29-51, 54-58, 60.

- (a) Elliptic cylinder parallel to y -axis.
(b) Parabolic cylinder parallel to x -axis.
- (a) xy : hyperbola, xz : hyperbola, yz : no solution. Hyperboloid of two sheets.
(b) xy : hyperbola, xz : hyperbola, yz : ellipse. Hyperboloid of one sheet.
(c) $xy : y = \pm 3x$. $xz : (0, 0)$, $yz : y = \pm 3z$. Cone.

15.1) Graphs and Level Curves. 25-33, 34, 35, 36-43, 74-77.

- (a) plane.
(b) elliptic paraboloid.
(c) hyperbolic paraboloid.
- (a) lines.
(b) $c = 0, y = \pm x$. $c = -1, 1, 2$, hyperbolas.
(c) $c = 2, (0, 0)$. $c = -1, 0, 1$ circles.

15.2) Limits and Continuity. 10-12, 13-27, 29-34, 35-50, 52-53, 62-67, 71.

- (a) Function is continuous at $(-1, 2)$ so $\lim_{(x,y) \rightarrow (-1,2)} f(x, y) = f(-1, 2) = 0$.
(b) $\lim_{(x,y) \rightarrow (3,-1)} \frac{(x+3y)(x-3y)}{y(x+3y)} = \lim_{(x,y) \rightarrow (3,-1)} \frac{x-3y}{y} = -6$.

2. (a) $\lim_{(t,0) \rightarrow (0,0)} f(x, y) = \frac{1}{2}$, $\lim_{(0,t) \rightarrow (0,0)} f(x, y) = 2$.
 (b) $\lim_{(t,0) \rightarrow (0,0)} f(x, y) = 0$, $\lim_{(t,t) \rightarrow (0,0)} f(x, y) = \frac{1}{2}$.

15.3) Partial Derivatives. 1-9, 11-14, 15-30, 32-34, 38-46, 48-53, 54-59.

1. (a) $\lim_{h \rightarrow 0} \frac{(3(x+h)-y)-(3x-y)}{h} = 3$, $\lim_{h \rightarrow 0} \frac{(3x-(y+h)-(3x-y))}{h} = -1$.
 (b) $\lim_{h \rightarrow 0} \frac{(x+h)y^2 - xy^2}{h} = y^2$, $\lim_{h \rightarrow 0} \frac{x(y+h)^2 - xy^2}{h} = 2xy$.
2. (a) $f_{xx} = -y^2 \sin(xy)$, $f_{xy} = \cos(xy) - xy \sin(xy)$, $f_{yx} = \cos(xy) - xy \sin(xy)$, $f_{yy} = -x^2 \sin(xy)$.
 (b) $f_{xx} = \frac{2(y^3 - x^2)}{(x^2 + y^3)^2}$, $f_{xy} = f_{yx} = \frac{-6xy^2}{(x^2 + y^3)^2}$, $f_{yy} = \frac{6x^2y - 3y^4}{(x^2 + y^3)^2}$

15.4) The Chain Rule. 9-18, 19-26, 27-28, 29-30, 35-40, 57-59, 65, 67-69, 72-73, 75.

1. $\frac{dw}{dt} = 3x^2y^2(2t) + 2x^3y(1 + e^{2-t}) = 3(25)(1)(4) + 2(5^3)(1)(2) = 800$.
2. $\frac{\partial w}{\partial u} = \frac{1}{x}(2) - \frac{1}{y} \frac{v^2}{u^2} = 2 - \frac{1}{2} = \frac{3}{2}$.
 $\frac{\partial w}{\partial v} = \frac{1}{x}(3) + \frac{1}{y} \frac{2v}{u} = 3 - 2 = 1$.
3. (a) $\frac{\partial F}{\partial x}(0) + \frac{\partial F}{\partial y} \left(\frac{\partial y}{\partial z} \right) + \frac{\partial F}{\partial z}(1) = 0 \Rightarrow \frac{\partial y}{\partial z} = -\frac{\partial F}{\partial z} / \frac{\partial F}{\partial y}$.
 (b) $\frac{\partial y}{\partial z} = \frac{-xye^{xz} - xye^{yz}}{e^{xz} + xze^{yz}}$
4. $\frac{\partial z}{\partial x} = \frac{-2xy - z^3}{y^2 + 3xz^2}$, $\frac{\partial z}{\partial y} = \frac{-2yz}{y^2 + 3xz^2}$.
5. $\frac{dE}{dt} = \frac{1}{2} 2v \frac{dv}{dt} + g \sin(\theta) \frac{d\theta}{dt} = -g \sin(\theta)v + g \sin(\theta)v = 0$.

15.5) Directional Derivatives and Gradient. 1-10, 11-12. 13-20, 21-30, 31-36, 43-44, 47-50, 59-64, 69-72, 74, 75-78, 81, 82, 85.

1. $(\nabla f)(-1, 2, 1) = (-4, 1, 3)$. $(D_{\vec{u}}f)(-1, 2, 1) = \frac{8}{3} + \frac{2}{3} + 1$.
2. (a) $\sqrt{145}$, $\langle \frac{8}{\sqrt{145}}, \frac{-9}{\sqrt{145}} \rangle$.
 (b) $\langle 9, 8 \rangle$.
3. $f(x, y) = xy$.
 (a) Graph the level set of f through the point $(2, 1)$.
 (b) Include the vector $\nabla f(2, 1)$ on your graph.
 (c) $(x - 2) + 2(y - 1) = 0 \Rightarrow y = -\frac{1}{2}x + 2$.
 (d) The vector $\nabla f(2, 1)$ is perpendicular to the level set of f at $(2, 1)$.

15.6) Tangent Planes. 3-4, 9, 11, 13-28, 29-32, 54-56.

1. Find the tangent plane to the surface at the given point.

(a) $-4(x + 1) - 1(x - 2) + 4(x - 2) = 0$.

(b) $1(x - 3) + 6(y - 0) + 0(z - 2) = 0$ or $x + 6y = 3$.

(c) $\frac{1}{2}(x - 3) - 2(y - 1) - \frac{1}{2}(z + 1) = 0$.

2. Find the tangent plane to the surface at the given point.

(a) $\frac{5}{3}(x - 5) - \frac{4}{3}(x - 4) - (z - 3) = 0$.

(b) $0(x - 2) + 2(y - 0) + 1(z - 3) = 0$ or $2y - 3z = 3$.

(c) $3(x - 2) - (y - 1) + 4(z + 1) = 0$.

15.7) Maximum/Minimum Problems. 9-12, 13-22, 23-37, 41-42, 43-46, 62-66, 71.

1. Find any critical points and classify each as relative maximum, relative minimum, or saddle.

(a) Local max at $(0, 0)$.

(b) local min at $(0, 1)$

(c) Saddle at $(1, 2)$.

(d) Local min $(\pm 1, 0)$. Local max $(0, 0)$.

2. $\min x^2 + y^2 + z^2 = 2$ at $(x, y) = (1, 0)$.

3. $\max V = 2$ at $(x, y) = (1, 1)$.

15.8) Lagrange Multipliers. 3-4, 5-6, 7-23, 26, 27-36.

1. $\max = 3\sqrt{3}$, $\min = -3\sqrt{3}$.

2. Area = 32.

3. $D = \sqrt{30}$.

16.1) Double Integrals, Rectangular Regions. 1-3, 5-6, 7-24, 25-35, 36-39, 40-45, 46-50, 53-54.

1. $\frac{1}{4} \int_0^2 \int_1^3 2y - x \, dy \, dx = 3$.

2. (a) $\int_0^{\ln(3)} \int_0^1 xy e^{xy^2} \, dy \, dx = 1 - \frac{1}{2} \ln(3)$.

$$(b) \int_0^3 \int_0^1 \frac{y}{\sqrt{1+xy}} dx dy = \frac{10}{3}.$$

16.2) Double Integrals, General Regions. 5-8, 9-10, 11-27, 28-34, 35-42, 43-53, 57-62, 63-68, 69, 70, 71, 73-80, 85-90, 95-96, 99-102.

$$1. \int_1^2 \int_{y^2+3}^{3y+1} 2y - 1 dx dy = \frac{1}{3}.$$

$$2. (a) \int_0^3 \int_0^{2y} e^{y^2} dx dy = e^9 - 1.$$

$$(b) \int_{-2}^2 \int_{x^2}^4 e^{12x-x^3} dy dx = \frac{1}{3}(e^{16} - e^{-16})$$

16.3) Double Integrals in Polar Coordinates. 7-10, 11-14, 15-18, 19-20, 21-30, 31-40, 42, 44-46, 47, 49-50, 53-54, 57-60, 65-68.

1. Evaluate the integral by converting to polar.

$$(a) \int_0^4 \int_0^{\frac{\pi}{2}} (r \cos(\theta))(r \sin(\theta))r dr d\theta = 32.$$

$$(b) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{1}{1+r^2}}^1 r dr d\theta = \frac{2\pi}{5}.$$

$$(c) \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{\sec(\theta)}^{2\sec(\theta)} r dr d\theta = 3.$$

16.4) Triple Integrals. 4-6, 7-14, 15-29, 30-35, 36-37, 38-46, 47-50, 51-54, 57-58, 62-63, 67-70.

$$1. \int_0^2 \int_0^{1-\frac{1}{4}x^2} \int_0^{4-4y-x^2} dz dy dx$$

$$\int_0^1 \int_0^{\sqrt{4-4y}} \int_0^{4-4y-x^2} dz dx dy$$

$$\int_0^2 \int_0^{4-x^2} \int_0^{1-\frac{1}{4}z-\frac{1}{4}x^2} dy dz dx$$

$$\int_0^4 \int_0^{\sqrt{4-z}} \int_0^{1-\frac{1}{4}z-\frac{1}{4}x^2} dy dx dz$$

$$\int_0^4 \int_0^{1-\frac{1}{4}z} \int_{\sqrt{4-4y-z}} dx dy dz$$

$$\int_0^1 \int_0^{4-4y} \int_{\sqrt{4-4y-z}} dx dz dy$$

$$2. (a) \int_0^2 \int_{-\sqrt{4-2z}}^{\sqrt{4-2z}} \int_{x^2}^{4-2z} dy dx dz = \frac{256}{30}.$$

$$(b) \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{2-\frac{1}{2}y} dz dx dy = \frac{256}{30}.$$

16.5) Cylindrical and Spherical Coordinates. 3-4, 9-10, 11-14, 15-22, 23-28, 29-34, 35-38, 41-47, 48-54, 58-61, 62-63, 64-65, 66-72, 77-79.

$$1. \int_0^{2\pi} \int_1^2 \int_0^{4-r^2} \frac{z}{r^{\frac{3}{2}}} r dz dr d\theta = \frac{7}{3}\pi.$$

$$2. \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^4 \cos(\theta) \sin^3(\phi) \cos(\phi) d\rho d\phi d\theta = \frac{32}{15}.$$

3. (a) $\int_0^{2\pi} \int_0^3 \int_r^3 r \, dz \, dr \, d\theta = 9\pi.$
 (b) $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{3\sec(\phi)} \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta = 9\pi.$
4. (a) $\int_0^{2\pi} \int_3^5 \int_0^{\sqrt{25-r^2}} r \, dz \, dr \, d\theta = \frac{128\pi}{3}.$
 (b) $\int_0^{2\pi} \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \int_{3\csc(\phi)}^5 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta = \frac{128\pi}{3}.$

16.7) Change of Variables. 5-11, 13-16, 17-22, 23-26, 27-30, 31-36, 37-39, 41-44, 46-47, 48, 50-52, 53, 56.

1. $\int_1^2 \int_1^3 \frac{1}{2v} \, dv \, du = \frac{1}{2} \ln(3).$
 2. $\int_0^2 \int_1^3 (v - 2u) \left(\frac{1}{2}\right) \, dudv = -6.$
 3. $\int_0^\pi \int_0^1 10r^2 \sin(\theta) \, dr \, d\theta = \frac{20}{3}.$

17.1) Vector Fields. 2, 8-15, 18, 24, 25-30, 35-42, 43-45, 47-48, 49-52.

1.
2.

17.2) Line Integrals. 4-10, 12-16, 17-34, 35-36, 39-40, 41-46, 47-48, 49-56, 57-60, 62, 64-65, 68, 70-72, 73.

1. Evaluate the line integrals.
 (a) $\int_0^2 t^3(0) + t(2t) \, dt = \frac{16}{3}.$
 (b) $\int_0^1 (2t)(-1 + 2t)(3) \, dt = 1.$
 (c) $\int_{-1}^2 (t^4 - t)(4t^3) + 2t \, dt = 104.1.$

17.3) Conservative Vector Fields. 7, 9-16, 17-30, 31-34, 39-42, 44, 45-50, 51-52, 54-56, 59-62, 63-64.

1. (a) $\phi(x, y) = \sqrt{x^2 + y^2}.$
 (b) $\phi(3, 4) - \phi(1, 0) = 4.$
2. $\vec{F} = \langle 1 + \cos(y), 2y - x \sin(y) \rangle.$
 (a) $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y} = -\sin(x).$
 (b) $\phi(x, y) = x + x \cos(y) + y^2.$
 (c) $\phi(2, \pi) - \phi(1, 0) = \pi^2 - 2.$

3. $\nabla \times \vec{F} = \vec{0}, \phi(x, y, z) = x^2 + xy + 2y + z^3.$

17.4) Green's Theorem. 9-14, 15-16, 17-20, 21-25, 27-30, 31-40, 41-45, 48, 53-54.

1. $\int_0^2 \int_0^1 2x - 2x \, dydx = 0.$

2. (a) $\int_{-1}^1 t - t^2 + (t + t^2)(2t) \, dt + \int_0^1 (1 - 2t - 1)(-2) \, dt = \frac{2}{3} + 2 = \frac{8}{3}.$

(b) $\int_{-1}^1 \int_{x^2}^1 2 \, dydx = \frac{8}{3}.$

3. $\int_0^1 \int_0^{2x} 4 \, dydx = 4.$

4. $\int_0^{2\pi} 16 \cos^4(t) + 16 \sin^4(t) \, dt = 24\pi, \int_0^{2\pi} \int_0^2 3r^3 \, drd\theta = 24\pi. (The \textit{first} \textit{integral} \textit{is} \textit{a} \textit{little} \textit{too} \textit{hard} / \textit{tedious})$

17.5) Divergence and Curl. 9-16, 17, 19, 21-22, 27-34, 41, 42, 44, 65, 67-69, 73.

1. \vec{F} should be $\langle x^3, x^2y, yz^3 \rangle. \nabla \cdot \vec{F} = 4x^2 + 3yz^2, \nabla \times \vec{F} = \langle z^3, 0, 2xy \rangle.$

2. $\nabla \times \langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \rangle = \vec{0}.$

3. $\nabla \times \vec{F} = \langle -xy, -ze^{xz}, yz - 2y \rangle \neq \vec{0}.$

17.6) Surface Integrals. 9-14, 15, 17-18, 19-24, 25-28, 29-34*, 35-38*, 43-48*, 52, 70-72, 74-75.

*: For 29-34, 35-38, 43-48, you **must** parametrize the surface by $r(u, v)$ and use the parametric form to do the integrals, rather than the 'explicit' form indicated in the book here.

1. $\int_0^{2\pi} \int_0^1 \sqrt{1 + r^2} r \, drd\theta = \frac{\pi}{6}(5^{\frac{3}{2}} - 1)$

2. $\int_0^{2\pi} \int_0^2 (-\cos(\theta) + 2)r \, drd\theta = 8\pi$

3. $\int_0^3 \int_{u^2}^{3u} \sqrt{6} \, dvdu = 4.5\sqrt{6}$

4. $d\vec{S} = \langle -f(v)f'(v), f(v)\cos(u), f(v)\sin(u) \rangle \, dudv, dS = f(v)\sqrt{1 + (f'(v))^2} \, dudv$

5. (a) $\int_0^\pi \int_0^1 \int u \cos(v) \sin(v) - u \cos(v) \sin(v) + uv \, dudv = \frac{\pi^2}{4}.$

(b) $\int_0^\pi \int_0^1 u \sin(v) \sqrt{1 + u^2} \, dudv = \frac{2}{3}(2^{\frac{3}{2}} - 1).$

17.7) Stokes' Theorem. 5-10, 11-16, 17-24, 30-33, 45.

1. $\int_0^{2\pi} 4 \sin^2(t) + 4 \cos^2(t) dt = 8\pi, \int_0^{2\pi} \int_0^2 2r dr d\theta = 8\pi.$
2. $\int_0^{2\pi} 9 \sin^2(t) + 9 \cos^2(t) dt = 18\pi.$
3. $\int_0^1 \int_0^{1-u} \int 2 dv du = 1$

17.8) Divergence Theorem. 9-12, 13-16, 17-24, 25-27, 30.

1. $\int_0^{2\pi} \int_0^2 \int_{r^2}^2 r dz dr d\theta = \frac{128\pi}{3}. - \int_0^{2\pi} \int_0^2 r^5 dr d\theta + \int_0^{2\pi} \int_0^2 16r dr d\theta = \frac{128\pi}{3}.$
2. Use the divergence theorem to evaluate $\iint_S \vec{F} \cdot d\vec{S}$. The surfaces have outward pointing normal.
 - (a) $\int_0^2 \int_0^{3-1.5z} \int_0^{6-2y-3z} 3 dx dy dz = 18.$
 - (b) $\int_0^{2\pi} \int_0^\pi \int_0^3 3\rho^5 \sin^3(\phi) \cos(\phi) d\rho d\phi d\theta = 0.$