1. Define $h(x) := f(x) - g(x)$ on $[a, b]$.
   i. Since $f$ and $g$ are Riemann integrable on $[a, b]$, $h$ is Riemann integrable on $[a, b]$.
   ii. $f = g$ if $x \neq c \Rightarrow h(x) = 0$ if $x \neq c$, and the only pt where $h \neq 0$ is at $x = c$.
   $\therefore h$ is continuous on $[a, b]$ w/ only one pt of discontinuity (removable) at $x = c$.
   iii. $\Rightarrow h$ is Riemann integrable on $[a, b]$ and the measure of the set of all discontinuities of $h$ is zero.
   iv. $\therefore \int_a^b h = \int_a^c h + \int_c^b h = 0 \Rightarrow \int_a^b (f-g) = 0 \quad \Leftrightarrow \quad \int_a^b f = \int_a^b g$.

b) Define $h := f-g$ on $[a, b]$.
   i. $h$ is Riemann integrable on $[a, b]$ since $f$ and $g$ are.
   ii. $h(x) = 0$ if $x \neq c_1, \ldots, c_n$.
   $\Rightarrow h$ is continuous on $[a, b]$ except for n removable discontinuities at $c_1, \ldots, c_n$.
   $\Rightarrow m(\{\text{set of all discontinuities of } h\}) = 0$.
   iii. $\int_a^b h = \int_a^{c_1} h + \int_{c_1}^{c_2} h + \ldots + \int_{c_{n-1}}^{c_n} h = 0$, assuming w.l.o.g. that $c_1 < c_2 < \ldots < c_n$, and using $h \equiv 0$ if $x \neq c_1, \ldots, c_n$.
   iv. $0 = \int_a^b h = \int_a^b (f-g)$.

c) Note that if we don't read this question carefully, we may want to answer "yes" to this question (since we may think $h := f-g \equiv 0$ if $x \neq c_1, c_2, \ldots$ and $m(\{c_1, \ldots\}) = 0$). However, note that $g$ is assumed to be only bdd and thus not necessarily Riemann integrable. For example, let $g(x) := \begin{cases} 1, & x \text{ rational} \\ -1, & x \text{ irrational} \end{cases}$ on $[0, 1]$.

Then, $g$ is bdd but not integrable on $[0, 1]$ w/ $\int (g) = 1$ and $\int (g) = -1$. 
Let $f$ and $I_i$ be as described.

i. Note that the right-hand and left-hand limits of $f$ at the endpoints of $I_i$ exist. Since $f = c_i$ on $I_i$, this implies that $f$ has jump discontinuities at the endpoints of $I_i$.

ii. $\because m(\{\text{set of discontinuities of } f\}) = 0$, since this is a finite set.

$\therefore$ By the Riemann-Lebesgue Thm $f$ is Riemann integrable.

iii. Hence, we may write:

\[
\int_a^b f = \int_{I_1}^{b_1} f = \int_{I_1}^{b_1} c_1 = \int_{I_1}^{b_1} c_1 \, dx = \sum_{i=1}^n c_i (b-a)
\]

since $\sum_{i=1}^n m(I_i) = m([a, b]) = b-a$.

10. Let $f$ be as defined by the graph, i.e., a step function with a jump discontinuity at $c$.

Claim: let $F(x) := \int_a^x f$ on $[a, b]$.

Then, $F$ is not differentiable at $x=c$.

Pf.

i. Since $m(\{\text{set of discontinuities of } f\}) = 0 \Rightarrow f$ is Riemann integrable on $[a, b]$.

$\therefore$ $\int_a^b f$ is well-defined $\Rightarrow F$ is well-defined.

ii. Since $f$ is bdd on $[a, b]$, $F$ is continuous on $[a, b]$.

iii. By def, if $F'(c)$ exists, it would have to be

\[
\lim_{h \to 0} \frac{F(c+h) - F(c)}{h}
\]

Here, we show that this $\lim$ is DNE.

iv. Consider the right-hand limit.

\[
\lim_{h \to 0^+} \frac{F(c+h) - F(c)}{h} = \lim_{h \to 0^+} \frac{1}{h} \left( \int_a^{c+h} f - \int_a^c f \right) = \lim_{h \to 0^+} \frac{1}{h} \int_c^{c+h} f
\]

\[
= \lim_{h \to 0^+} \frac{1}{h} \cdot k_2 \cdot m(\{c, c+h\}) = \lim_{h \to 0^+} \frac{1}{h} \cdot k_2 \cdot h
\]

\[
= k_2
\]
v. Consider the left-hand limit:

\[
\lim_{h \to 0^-} \frac{F(c+h) - F(c)}{h} = \lim_{h \to 0^-} \frac{1}{h} \left( \int_{c+h}^{c} f(x) \, dx - \int_{c}^{c} f(x) \, dx \right) = \lim_{h \to 0^-} \frac{1}{h} \int_{c}^{c+h} f(x) \, dx
\]

\[
= \lim_{h \to 0^-} \frac{-1}{h} \int_{c+h}^{c} f(x) \, dx = \lim_{h \to 0^-} \frac{-1}{h} \cdot k_1 \cdot m(c, c+h)
\]

\[
= \lim_{h \to 0^-} \frac{-1}{h} \cdot k_1 \cdot (-h) = k_1
\]

vi. Since \( k_1 \neq k_2 \), \( F \) is not differentiable at \( c \).

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13) a) \( f(x) = x^2 + 1 \);

\[
f(c)(x-c) = \int_{c}^{x} f(t) \, dt \iff f(c), y = (\frac{x^3}{3} + x) \bigg|_{c}^{x} \iff f(c) = \frac{1}{4}(\frac{208}{3} + y) = \frac{55}{3}
\]

\[
\Rightarrow c^2 + 1 = \frac{55}{3} \Rightarrow c^2 = \frac{52}{3} \Rightarrow c = \frac{2\sqrt{13}}{3}
\]

b) Let \( f \) be defined by the graph. Then, \( f \) is Riemann-integrable on \([1,3]\)

and \( \int_{1}^{3} f = (1)(2-1) + (2)(3-2) = 3 \)

\[
\Rightarrow \frac{1}{b-a} \int_{a}^{b} f = \frac{1}{3-1} \int_{1}^{3} f = \frac{3}{2}
\]

However, \( \exists c \in (1,3) \) s.t. \( f(c) = \frac{3}{2} \).