5. a. Assume that \( \{a_n\} \) has infinitely many terms > \( L + \varepsilon \), for some \( \varepsilon > 0 \).

   i) Let \( M_1 > L + \varepsilon \). Then, either \((L + \varepsilon, M_1)\) or \((M_1, \infty)\) contain infinitely many terms of \( \{a_n\} \). WLOG assume \((M_1, \infty)\) contains infinitely many terms of \( \{a_n\} \). Choose \( a_n \in (M_1, \infty) \).

   ii) Let \( M_2 > M_1 \). Then, either \((M_1, M_2)\) or \((M_2, \infty)\) contain infinitely many terms of \( \{a_n\} \). WLOG assume \((M_1, M_2)\) contains \( \omega \)-many terms of \( \{a_n\} \). Choose \( a_n \in (M_1, M_2) \) s.t. \( a_{n_2} \neq a_{n_1} \). Note that this is always possible.

   iii) Continue indefinitely. Then, \( \{a_n\} \) has the appropriate properties.

b. By Thm 2-14, every bounded seq has a convergent subseq. Hence, \exists a subseq of \( \{a_n\} \) that converges to a number \( K \) s.t. \( a_{n_k} > L + \varepsilon \) by construction, \( K > L + \varepsilon \) using previous results (e.g. #8, p.47).
Show that if every open interval containing $L$ contains a pt of the set $A$ distinct from $L$, then $\forall \varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ contains infinitely many pts of $A$.

**Pf by Contradiction:**

i) Assume for some $\varepsilon > 0$, $(L-\varepsilon, L+\varepsilon)$ contains only finitely many pts of $A$ distinct from $L$, e.g., $x_1, \ldots, x_n$.

ii) $\hat{\varepsilon} = \min_{i} |L - x_i|$,

iii) Consider $(L-\hat{\varepsilon}, L+\hat{\varepsilon})$. This interval does not contain any pt of $A$ distinct from $L$. $\Rightarrow \Leftarrow$

13. Let $\{a_n\}$ be a Cauchy seq.

a. Show $\{a_n\}$ is bdd.

**Pf:**

i) $\{a_n\}$ Cauchy seq. $\Rightarrow \forall \varepsilon > 0$, $\exists N(\varepsilon) > 0$ : $n, m > N \Rightarrow |a_n - a_m| < \varepsilon$.

ii) Let $\varepsilon = 1$. Then, $\exists N(1) > 0$ : $n, m > N(1) \Rightarrow |a_n - a_m| < 1$.

This implies that $\forall n, m > N(1)$, the terms of the seq. are bdd by $\{a_n, n > N(1)\}$.

iii) $\exists$ finitely many terms of the seq. before $a_{N(1)}$. Hence, they have a max, i.e., $K = \max \{a_i \mid 1 \leq i \leq N(1)\}$.

iv) Let $M = \max \{|a_{N(1)} + 1, K| \}$. Then, $|a_n| \leq M$, $\forall n$.

b. By Thm 2-14, every bdd seq. has at least one convergent subseq.

c. Pf by Contradiction:

i) Assume that $\{a_n\}$ has two distinct subseq. $\lim_{n\to\infty} p_n = L$ and $L_2$.

w.l.o.g. assume $L_2 > L_1$, and let $\varepsilon = \frac{L_2 - L_1}{2} > 0$.

w.l.o.g. assume $L_1$ is the lim of $\{a_{n_k}\}$, and $L_2$ the lim of $\{b_{n_k}\}$.
ii) By def \( \forall \varepsilon > 0, \exists N_1(\varepsilon) > 0 : n > N_1 \Rightarrow |a_n - L_1| < \frac{\varepsilon}{2} \), and
\[ \forall \varepsilon > 0, \exists N_2(\varepsilon) > 0 : n > N_2 \Rightarrow |b_n - L_2| < \frac{\varepsilon}{2}. \]

iii) Let, \( N := \max(N_1, N_2) \) for the selected \( \varepsilon \). Then, \( \forall n > N \),
\[ |a_n - b_n| > \varepsilon \] which contradicts the Cauchy criterion. \( \Rightarrow \varepsilon \)

d. By Thm 2-16, a seq converges iff it is bdd and has exactly one limit pt.

14. Show that a convergent seq is a Cauchy seq.

PF.
i) Let \( \{a_n\} \) be a convergent seq, and assume \( L \) is its limit. Thus,
\[ \forall \varepsilon > 0, \exists N(\varepsilon) > 0 : n > N \Rightarrow |a_n - L| < \varepsilon. \]

ii) Consider,
\[ |a_n - a_m| = |(a_n - L) + (L - a_m)| \leq |a_n - L| + |a_m - L|, \] by the \( \Delta \) inequality
\[ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \] by choosing \( N(\varepsilon) \)
\[ = \varepsilon \] sufficiently large.