4. Assume \( a > 1 \).
   a. By contradiction. Assume \( a^n < 1 \), \( \forall n \). Then, \( (a^n)^n < 1^n \iff a < 1 \).
   b. \( \frac{1}{n} > \frac{1}{n+1} \iff a^n > a^{n+1} \).
   c. NTS: \( \forall e > 0 \), \( \exists N(e) > 0 \) s.t. \( n > N \implies |a^n - 1| < e \).
      \( \iff -e < a^n - 1 < e \iff 1 - e < a^n < 1 + e \)
      \( \iff \frac{\ln(1-e)}{\ln a} < \frac{1}{n} < \frac{\ln(1+e)}{\ln a} \iff n > \frac{\ln a}{\ln(1+e)} \).
      \( \therefore \forall e > 0 \), choose \( N(e) = \frac{\ln a}{\ln(1+e)} \).

6. \( a_{n+1} = \sqrt{2 + a_n} \) and \( a_1 = \sqrt{2} \).
   a. Show \( a_n \leq 2 \), \( \forall n \).
      \underline{PF by induction:} Consider the statement \( P(n) : \{ a_n \leq 2, \forall n \in \mathbb{N} \text{ where } a_{n+1} = \sqrt{2 + a_n} \text{ with } a_1 = \sqrt{2} \} \).
      \underline{Step 1:} Show \( P(1) \) is true.
      \( a_2 = \sqrt{2 + a_1} = \sqrt{2 + \sqrt{2}} \leq 2 \). \( \checkmark \)
      \underline{Step 2:} Assume \( P(K) \) is true, i.e., \( a_{K+1} \leq 2 \).
      \underline{Step 3:} Show \( P(K+1) \) is true.
      \( P(K+1) : \ a_{K+2} = \sqrt{2 + a_{K+1}} \leq \sqrt{2 + 2} = 2 \). \( \checkmark \)
      \underline{Step 4:} \( \therefore P(n) \) is true \( \forall n \in \mathbb{N} \) by the PMI.
   b. Show \( \{ a_n \} \) is an increasing seq., and thus it converges.
      \underline{PF by induction:} Consider the statement \( P(n) : \{ a_{n+1} > a_n, \forall n \in \mathbb{N} \text{ where } a_{n+1} = \sqrt{2 + a_n} \text{ with } a_1 = \sqrt{2} \} \).
      \underline{Step 1:} Show \( P(1) \) is true.
      \( P(1) : \ a_2 = \sqrt{2 + a_1} = \sqrt{2 + \sqrt{2}} > \sqrt{2} = a_1 \). \( \checkmark \)
      \underline{Step 2:} Assume \( P(K) \) is true, i.e., \( a_{K+1} > a_K \).
Step 3: Show P(k+1) is true.

\[ P(k+1): a_{k+1} = \sqrt{2 + a_k} > \sqrt{2 + a_{k+1}} = a_{k+1}. \]

Step 4: \( P(n) \) is true \( \forall n \in \mathbb{N} \) by the PMI.

\( \Box \) Note that here we actually proved that \( \{a_n\} \) is strictly monotonically increasing.

\( \Box \) By part (a), the seq \( \{a_n\} \) is bounded above by 2. Here we proved that \( \{a_n\} \) is monotonically increasing. \( \therefore \) By Thm 2-6, \( \{a_n\} \) is convergent.

C. Show that \( \lim a_n = 2 \).

Pf:

i) In part (b) we proved that \( \{a_n\} \) is strictly monotonically increasing, bounded above by 2, and it is convergent.

ii) Assume \( \lim a_n = L \). Then from (i) it follows that \( L \leq 2 \) and \( L = \operatorname{l.u.b.} R(a_n) \).

iii) Claim: \( L = 2 \).

Pf. by Contradiction: Assume \( L < 2 \). Then

\[ L - \epsilon < a_n < L, \quad \text{since} \quad L = \operatorname{l.u.b.} R(a_n). \]

\[ \forall \epsilon > 0, \exists a_n \in \{a_n\}: \quad L - \epsilon < a_n < L, \quad \text{since} \quad L = \operatorname{l.u.b.} R(a_n). \]

Let, \( \epsilon = 2 + L - L^2 \). Note that \( \epsilon > 0 \) since this quadratic is positive on \( (-1, 2) \).

Simply factor \( 2 + L - L^2 \) and do a sign pattern. The intuition for this choice of \( \epsilon \) comes from the inequality \( L \leq \sqrt{2 + L} \).

Then, \( a_n > L - (2 + L^2) > L^2 - 2 \) \( \Rightarrow a_n + 2 > L^2 \)

\[ \Rightarrow \sqrt{a_n + 2} > L \quad \Rightarrow \quad a_{n+1} > L. \]

This is a contradiction since \( \{a_n\} \) is strictly monotonically increasing to 2.
10. Let \( \{a_n\} \) have the property: \( |a_{n+1} - a_n| < b^n \), for some \( 0 < b < 1 \). Show \( \{a_n\} \) is a Cauchy seq.

pf.

i) NTS: \( \forall \varepsilon > 0 \), \( \exists N : \forall m, n > N \Rightarrow |a_m - a_n| < \varepsilon \).

ii) W.L.O.G. assume \( m = n + k \) for some \( k \in \mathbb{N} \).

iii) Let \( \varepsilon > 0 \) be given. Then,

\[
|a_m - a_n| = |a_{n+k} - a_n| = \left| \sum_{j=0}^{k-1} (a_{n+j+1} - a_{n+j}) \right|
\]

\[
\leq \sum_{j=0}^{k-1} |a_{n+j+1} - a_{n+j}| \quad \text{by the } \Delta \text{ inequality}
\]

\[
\leq \sum_{j=0}^{k-1} b^{n+j} \quad \text{using the hypothesis}
\]

\[
= b^n \sum_{j=0}^{k-1} b^j
\]

\[
= b^n \cdot \frac{1 - b^k}{1 - b} \quad \text{a finite sum of a geometric series}
\]

\[
< \frac{b^n}{1 - b} \quad \text{since } 0 < b < 1, \quad 1 - b^k < 1.
\]

\[
< \varepsilon \quad \text{by choosing } n \text{ sufficiently large}
\]

using the Archimedean principle and \( 0 < b < 1 \). \( \blacksquare \)