4.5 Inverse of a Square Matrix

In this section, we will learn how to find an inverse of a square matrix (if it exists) and learn the definition of the identity matrix.

Identity Matrix for Multiplication:

- The number 1 is called the multiplicative identity for real numbers: $a(1) = a$

For example 5(1)=5

- In the set of *nxn* matrices we have a multiplicative inverse also: $AI=A=IA$
- *2x2* case

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$

Identity matrices

- **3 x 3 identity matrix**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \bullet \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

Inverse of a real number

- All real numbers (excluding 0) have an inverse. $a \cdot \frac{1}{a} = 1$

- For example $5 \cdot \frac{1}{5} = 1$

What about matrices?

- Some (not all) square matrices also have matrix inverses

$$\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =$$

Matrix Inverses

- Some (not all) square matrices also have matrix inverses
- If the inverse of a matrix A, exists, we call it A^{-1}
- Then, $A \cdot A^{-1} = A^{-1} \cdot A = I_n$

Inverse of a 2 x 2 matrix

- There is a simple procedure to find the inverse of a two by two matrix. This procedure **only works for the 2 x 2 case.**

An example will be used to illustrate the procedure:

A simple procedure to find the inverse of a **2 x 2** matrix.

- This procedure **only works for the 2 x 2 case.**
- Find the inverse of

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- *Det(A)* =difference of product of diagonal elements= $a_{11}a_{22} - a_{21}a_{12}$
- In order for the inverse of a 2 x 2 matrix to exist, Det(A) cannot equal to zero.
- **If** *Det(A)=0*, then we conclude the inverse does not exist and we stop all calculations.
- If *Det(A)* is non-zero then we proceed:

$$A^{-1} = \frac{1}{Det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Inverse of a two by two matrix

$$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

General procedure to find the inverse matrix

- We use a more general procedure to find the inverse of a 3 x 3 matrix.

Problem: Find the inverse of the matrix

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 2 \\ -2 & -2 & 1 \end{bmatrix}$$

Steps to find the inverse of any matrix

1. Augment this matrix with the 3 x 3 identity matrix.

2. Use elementary row operations to transform the matrix on the left side of the vertical line to the 3 x 3 identity matrix. The row operation is used for the **entire row** so that the matrix on the right hand side of the vertical line will also change.

3. When the matrix on the left is transformed to the 3 x 3 identity matrix, the matrix **on the right of the vertical line is the inverse.**

Procedure

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ -2 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3+2r_1=R_3]{r_2-2r_1=R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & -2 & 1 & 0 \\ 0 & -4 & 7 & 2 & 0 & 1 \end{array}\right]$$

$$\frac{1}{3}r_2 = R_2$$

TO NEXT LINE

Final result

The inverse matrix

- The inverse matrix appears on the right hand side of the vertical line and is displayed below.

$$\begin{bmatrix} 1 & -1 & -1 \\ -\frac{6}{5} & \frac{7}{5} & \frac{4}{5} \\ -\frac{2}{5} & \frac{4}{5} & \frac{3}{5} \end{bmatrix}$$