Warm up

Describe in words and shade on the number lines the solutions to:

$|x| < 3$

$|x-2| < 3$

$|x+2| < 3$

$|5x| < 15$

$|-5x| < 15$

Definition of limit of sequence of numbers

$e = 2.7182818284590452353602874713526624977572470936999595$

Roughly speaking the limit means that as n gets large $(1+\frac{1}{n})^n$ gets close to a finite number which we call e.

Rigorously using decimal expansion: $e = \lim_{n\to\infty}(1+\frac{1}{n})^n$ means that for every positive integer k there is a positive integer N such that: e and $(1+\frac{1}{n})^n$ agree up to k decimal places whenever $n > N$.

not using decimal expansion: means that for every real number $\varepsilon > 0$, there is an integer N such that $|e - (1+\frac{1}{n})^n| < \varepsilon$ whenever $n > N$.

n	$(1+1/n)^n$
1	2
2	9/4=2.25
10	2.5937…
100	2.7048…
1000	2.7169…
10000	2.7181…
100000	2.271826…
1000000	2.718280…

§10.1&3 Introduction to Limits

The student will learn about:

- **Functions and graphs**
- **limits from a graphic approach**
- **limits from an algebraic approach**
- **limits of difference quotients.**

Limits IMPORTANT!

This table shows what f (x) is doing as x approaches 2. Or we have the limit of the function as x approaches 2. Notation:

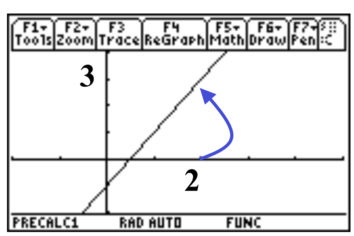
$$\lim_{x \to 2} 2x - 1 \;=\; 3$$

Idea: We write

$$\lim_{x \to c} f(x) = L$$

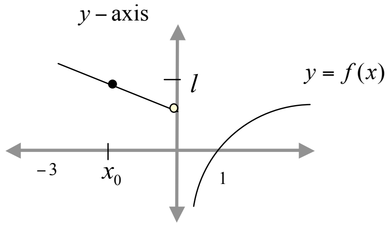
or as x → c; f (x) → L

if the functional value of f (x) is close to the single real number L whenever x <u>is close to, but not equal to, c</u>. (on either side of c).



x	1.5	1.9	1.99	1.999	2	2.001	2.01	2.1	2.5
f (x)	2	2.8	2.98	2.998	?	3.002	3.02	3.2	4

Limits of functions



Roughly speaking the $\lim_{x \to a} f(x) = l$ means that as x gets close to a $f(x)$ gets close to l.

Rigorously: Assume that $f(x)$ is defined on the open interval (a,b).

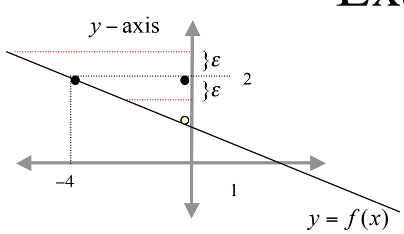
$\lim_{x \to a} f(x) = l$

means that for every real number $\varepsilon > 0$,

there is a real number $\delta > 0$ such that

$|f(x) - l| < \varepsilon$ whenever $|x - a| < \delta$.

Example:



$\lim_{x \to -4} f(x) = 2$.

For every real number $\varepsilon > 0$,

there is a real number $\delta > 0$ such that

$|f(x) - 2| < \varepsilon$ whenever $|x - (-4)| < \delta$.

1. Using the graph above compute $\lim_{x \to 1} f(x) =$

2. Using the graph of $y = 6$ compute $\lim_{x \to 1} 6 =$

3. Using the graph of $y = 2x$ compute $\lim_{x \to 2} 2x =$

Example:

4. Using the definition compute: $\lim_{x \to 3} 2x =$

How can we write an x that is close to 3?

$|f(x) - 6| = |f(3 + h) - 6| =$

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Example:

5. Using the definition compute: $\lim_{x \to 7} 4x - 5 =$

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Basic Properties of Limits:

Constant Function: $\lim_{x \to a} c =$ ______. Identity Function: $\lim_{x \to a} x =$ ______.

Assume that: $\lim_{x \to a} f(x) = b$ and $\lim_{x \to x_0} g(x) = d$

Sum Rule: $\lim_{x \to a} f(x) + g(x) =$

Product Rule: $\lim_{x \to a} \left(f(x)\ g(x) \right) =$ ____________________________.

Rational Function: Assume that g(x) is a polynomial and $g(x_0) \neq 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\qquad\qquad}{\lim_{x \to a} g(x)}$$

Powers of functions:

$$\lim_{x \to a} \left(f(x) \right)^r = \left(\lim_{x \to a} f(x) \right)^r$$

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Examples using properties:

$\lim_{x \to 2} 1 = 1.$

$\lim_{x \to 2} x^2 = 4.$

$\lim_{x \to 2} x^3 =$ ____________.

Show $\lim_{x \to 2} (x^3 + x^2 - 1) = 11.$

From this example we can see that for a polynomial function

$$\lim_{x \to c} f(x) = f(c)$$

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Examples using properties:

Find $\lim_{x\to 2}(3x^3 - 5x^2 + 6) =$

Find $\lim_{x\to 2}\dfrac{(x^3 + x^2 - 1)}{x - 5} =$

From this example we can see that for a rational function

$$\lim_{x\to c}\frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)} \text{ if } Q(c) \neq 0$$

Find $\lim_{x\to 5}\dfrac{2x - 1}{x - 5} =$

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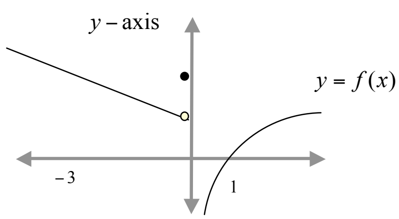
Limits from the left and from the right:



Roughly speaking the $\lim_{x\to x_0+} f(x) = l$

means that as x approaches x_0 from the right

________________________________.

In this example we have:

In order for a limit to exist, the limit from the left and the limit from the right must exist and be equal.

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Example

$$f(x) = \begin{cases} 2x \text{ if } x < 2 \\ -x + 8 \text{ if } 2 \le x \le 4 \\ x \text{ if } x > 4 \end{cases}$$

$\lim_{x \to 2^-} f(x) =$

$\lim_{x \to 2^+} f(x) =$

$\lim_{x \to 2} f(x) =$

$\lim_{x \to 4^-} f(x) =$

$\lim_{x \to 4^+} f(x) =$

$\lim_{x \to 4} f(x) =$

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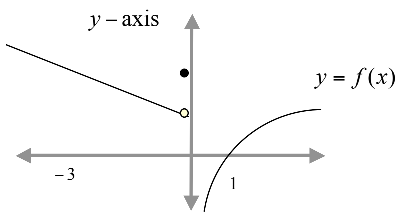
Where is the limit undefined?



$\lim_{x\to 0} f(x)$ does not equal a finite number

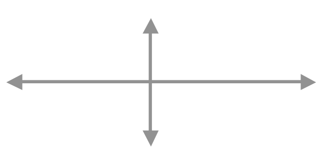
Roughly speaking:

No matter how close x gets to 0

____

What about $f(x) = \begin{cases} 1 \text{ if } x \text{ is rational} \\ -1 \text{ if } x \text{ is irrational} \end{cases}$

$\lim_{x\to 0} f(x) =$



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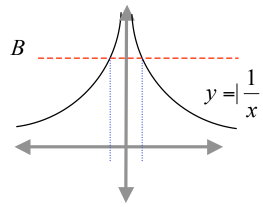
The sky is the limit! (limits that go to infinity)

Roughly speaking the $\lim_{x \to x_0} f(x) = \infty$ means that as x approaches x_0 ____________________________.



Rigorously: $\lim_{x \to a} f(x) = \infty$ means that for every real number $B > 0$, there is a real number $\delta > 0$ such that $|f(x)| > B$ whenever $|x - a| < \delta$.

Reciprocal Test : Let f be defined in an open interval about x_0, except possibly at x_0 itself. Then $\lim_{x \to x_0} f(x) = \infty$ if

1. For all x in some interval about x_0, $f(x)$ is positive; and
2. $\lim_{x \to x_0} \frac{1}{f(x)} = 0$.

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Indeterminate Form

If $\lim_{x \to c} f(x) = 0$ and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} \frac{f(x)}{g(x)}$ is said to be indeterminate.

The term indeterminate is used because the limit may or may not exist.

$$\lim_{x \to 5} \frac{(x^2 - 25)}{x - 5} =$$

$$\lim_{x \to 5} \frac{(x^2 - 25)}{(x - 5)^3} =$$

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