Integration Notes

Def: A partition of \([a, b]\) is a finite set of points \(P = \{x_0, \ldots, x_n\}\) where \(a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\).

Note: \(\Delta x_i = x_i - x_{i-1}\) & \(||P|| = \max_i \{\Delta x_i\}\)

Def: Suppose \(f\) is bounded on \([a, b]\) and \(P = \{x_0, \ldots, x_n\}\) be a partition of \([a, b]\).

- Let \(M_i(f) = \sup \{f(x): x \in [x_{i-1}, x_i]\}\) and
- Let \(m_i(f) = \inf \{f(x): x \in [x_{i-1}, x_i]\}\)
- Let \(\overline{S}(f; P) = \sum_{i=1}^n M_i(f) \Delta x_i\) be the upper sum
- Let \(\underline{S}(f; P) = \sum_{i=1}^n m_i(f) \Delta x_i\) be the lower sum.

Def: Let \(P\) & \(Q\) be partitions of \([a, b]\) \(Q\) is called a refinement of \(P\) i.e. \((P \subseteq Q)\)

Lemma: Suppose \(P\) & \(Q\) are partitions of \([a, b]\) with \(Q\) a refinement of \(P\) then.

- \(\overline{S}(f; Q) \leq \overline{S}(f; P)\)
- \(\underline{S}(f; Q) \geq \underline{S}(f; P)\)

Theorem: If \(P_1\) & \(P_2\) are any two partitions of \([a, b]\) then \(\underline{S}(f; P_1) \leq \overline{S}(f; P_2)\)

Def: Let \(\overline{S}(f) = \inf \{\overline{S}(f; P): P \text{ is a partition of } [a, b]\}\) & Let \(\underline{S}(f) = \sup \{\underline{S}(f; P): P \text{ is a partition of } [a, b]\}\)

Note: \(\underline{S}(f) \leq \overline{S}(f)\)

Def: Let \(f\) be bounded on \([a, b]\) we say \(f\) is Riemann integrable on \([a, b]\) if \(\underline{S}(f) = \overline{S}(f)\). In this case the common value is called the integral of \([a, b]\) and denoted \(\int_a^b f(x)\,dx\) or \(\int_a^b f\)

Theorem: (Riemann Condition for Integrability) A bounded function \(f\) on \([a, b]\) is Riemann integrable iff \(\forall \varepsilon > 0, \exists \text{ a partition } P(\varepsilon) \text{ of } [a, b]: \overline{S}(f; P) - \underline{S}(f; P) < \varepsilon\)

Theorem: If \(f\) is monotonic on \([a, b]\) then \(f\) is integrable on \([a, b]\).

Theorem: If \(f\) is continuous on \([a, b]\) then \(f\) is integrable on \([a, b]\).

Theorem: Suppose \(f\) is bounded on \([a, b]\) & continuous on \([a, b]\) except at some point \(c \in (a, b)\) then \(f\) is integrable on \([a, b]\).

Remark: Suppose \(f, g\) are bounded functions of a set \(A\) then the \(\sup\{f(x) + g(x): x \in A\} \leq \sup\{f(x): x \in A\} + \sup\{g(x): x \in A\}\)
Theorem: Suppose $f$ & $g$ are integrable on $[a, b]$. Then $f + g$ is integrable on $[a, b]$ & 
$$\int_a^b (f + g) = \int_a^b f + \int_a^b g$$

Theorem: Let $c \in (a, b)$ if $f$ is integrable on $[a, c]$ and $f$ is integralbe on $[c, b]$ then $f$ is integrable on $[a, b]$ and we say $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Note: $\int_a^b f = -\int_b^a f$ also $\int_a^a f = 0$.

Theorem: If $f$ is integrable on $[a, b]$, & $f(x) \geq 0, \forall x \in [a, b]$, then $\int_a^b f \geq 0$

Theorem: If $f$ & $g$ are integrable on $[a, b]$, & $f(x) \geq g(x), \forall x \in [a, b]$, then $\int_a^b f \geq \int_a^b g$

Theorem: If $f$ is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ & $|\int_a^b f| \leq \int_a^b |f|$

Theorem: If $f$ is integrable on $[a, b]$ & $m \leq f(x) \leq M, \forall x \in [a, b]$, then $m(b-a) \leq \int f \leq M(b-a)$

Theorem: (MVT for Integrals): Suppose $f$ is continuous on closed on $[a, b]$, then
$$\exists c \in [a, b]: \int_a^b f = f(c)(b-a)$$

Theorem: Suppose $f$ is bounded and integrable on $[a, b]$. Let $F(x) = \int_a^x f, \forall x \in [a, b]$, then $F(X)$ is a continuous function on $[a, b]$.

Theorem: (Fundamental Theorem of Calculus): Let $f$ be integrable on $[a, b]$ & $F$ is another function on $[a, b]$: $F$ satisfies the hypothesis of MVT & $F'(x) = f(x), \forall x \in (a, b), then \int_a^b f = F(b) - F(a)$