1. Show that the projective plane $P$ is the union of a Mobius strip and a disk.

2. Give two proofs that the Klein bottle $K$ is the union of two Mobius strips. (One proof should be a direct pictorial proof. The other follows from the fact that $K = P#P$.)

Solutions by Sarah Woolf.

2. Please see picture below (where + means $\cup$).

3. Proof 1: $K = P#P = (P - D^2) \cup_{\partial} (P - D^2) = M \cup M$. (The second equality is the definition of connected sum; the third is the previous problem.)

Proof 2: Please see the picture below.
3. Prove that $T\#P$ is homeomorphic to $K\#P$.

- **Solution by Ashar Ali.**

  First let us consider the following torus and the projective plane.

  ![Diagram of torus and projective plane](image)

  Now let us perform the connected sum to get $T\#P$.

  ![Connected sum diagram](image)

  Now let us cut along the indicated diagonal and let us name the new edge $e$.

  ![Cut along diagonal](image)

  After cutting along the diagonal $e$, let us reassemble along $a$.

  ![Reassembled diagram](image)
Now let us cut along this indicated diagonal $f$ and reassemble along $d$.

And now let us cut along the indicated diagonal.

And we have our Klein bottle connected summed with the projective plane.

- Solution by John Foss.
  A video illustrating this proof can be seen at http://www.csun.edu/ tf54692/pump.html. Begin with $T\#P$. We know that $P$ contains an embedded mobius strip. Send one side of the torus through the twist of the mobius strip. Continue until one end of the torus ends up inside the other. Disconnect the projective plane from this surface, and a Klein bottle will be left.
4. Prove that the surfaces $S^2$ and $\Sigma_n$ do not contain an embedded Mobius strip. (Hint: Suppose that a surface does contain Mobius strip. Observe that it has a triangulation which produces a polygon with a pair of edges of Type O.)

Solution by Elizabeth Leyton.

Proof: Consider a Mobius strip with a triangulation. Let this strip be embedded in a surface. This surface can be represented as a polygon with edges identified.

By following the steps of the proof of the Classification of Surfaces Theorem, I will arrive at the conclusion that this surface can only be homeomorphic to $\Pi_n$.

By step 1, the polygon with identified edges has edges that appear in pairs of two types: type S and type O. Pairs of edges of type S are of the form $aa^{-1}$ while as edges of type O are of the form $aa$. Because of the embedded Mobius strip, the polygon must have at least one pair of edges of type O.

By step 2, adjacent edges of type S can be eliminated. For example, $bb^{-1}$ can be eliminated from the polygon. Note that this step removes NO edges of type O.

By step 3, this polygon can be transformed to one in which all the vertices are the same point by cutting the polygon and reattaching along different edges. After this procedure, again adjacent edges of type S will be eliminated. Note again that this step removes NO edges of type O.

By step 4, this polygon can be transformed to one in which all the edges of type O are adjacent. After this procedure, again all adjacent edges of type S can be eliminated. Note again that this step removes NO edges of type O.

By step 5, this polygon can be transformed to one in which all the pairs of pairs of edges of type S can be made adjacent, i.e. all pairs of pairs of edges of type S occur following each other.

By step 6, any polygon with a mixture of edges of type S and type O may be rewritten as to only have adjacent edges of type O. However, if a polygon has edges of only type S it cannot be rewritten as to include edges of type O.

After all of these steps, we have eliminated all adjacent edges of type S, made all remaining pairs of pairs of edges of type S adjacent, and made all edges of type O adjacent. Since our polygon has a mixture of edges of type S and type O, it can be rewritten as to have only edges of type O. This means that our surface is a non-orientable surface and can therefore only be homeomorphic to $\Pi_n$.

Since $S^2$ and $\Sigma^2$ are not homeomorphic to $\Pi_n$, this means that the surfaces $S^2$ and $\Sigma^2$ do not contain a Mobius strip.
5. Prove that every compact, non-orientable surface is homeomorphic to either $\Sigma_n \# P$ or $\Sigma_n \# K$, for some $n \geq 0$.

(For this, define $\Sigma_0 = S^2$.)

Solution by Stacey Disbrow.

Proof. By Induction.

Recall from our Classification of Surfaces Theorem, which was proven in class, that every compact, non-orientable surface is homeomorphic to $\Pi_n$ for some $n$.

First, a few definitions:

Let us define:

- an odd non-orientable surface to be of the form $\Pi_i$, where $i$ is of the form $2n + 1$.
- an even non-orientable surface to be of the form $\Pi_j$ where $j$ is of the form $2n$.

Let us test a few cases in an attempt to find a pattern.

Case: $n = 1$ : Define $\Sigma_0 = S^2$, the sphere.

$\Pi_1 = P \cong S^2 \# P$ since $S^2$ is the identity

$\cong \Sigma_0 \# P$

Case: $n = 2$ :

$\Pi_2 = P \# P \cong K \cong S^2 \# K$

$\cong \Sigma_0 \# K$

Case: $n = 3$ :

$\Pi_3 = P \# P \# P \cong (P \# P) \# P \cong K \# P \cong T \# P$

$\cong \Sigma_1 \# P$

Case: $n = 4$ :

$\Pi_4 = P \# P \# P \# P \cong (P \# P) \# (P \# P) \cong K \# (P \# P) \cong (K \# P) \# P$

$\cong (T \# P) \# P \cong (\Sigma_1 \# P) \# P \cong \Sigma_1 \# (P \# P)$

$\cong \Sigma_1 \# K$

Notice, we begin to see a pattern. It appears that the odd non-orientable surfaces are homeomorphic to the $n$-holed torii connected sum the Projective Plane, and the even non-orientable surfaces are homeomorphic to the $(n - 1)$-holed torii connected sum the Klein Bottle.

Let us test this theory...

Claim: $\Pi_{2r+1} \cong \Sigma_r \# P$ and $\Pi_{2r} \cong \Sigma_{r-1} \# K$ for any $r \geq 0$.

Proof: by induction.

Assume for any $k \geq 0$, that is, assume that $\Pi_{2k+1} \cong \Sigma_k \# P$ and $\Pi_{2k} \cong \Sigma_{k-1} \# K$. 

Consider $\Pi_{2(k+1)+1} \cong \Pi_{2k+3} \cong \Pi_{2k} \# \Pi_3$  
$\cong (\Sigma_{k-1} \# K) \# (\Sigma_1 \# P)$ from our inductive assumption and case 3  
$\cong (\Sigma_{k-1} \# \Sigma_1) \# (K \# P)$ rearranging  
$\cong (\Sigma_k) \# (\Sigma_1 \# P)$ as seen in case 2  
$\cong (\Sigma_k \# \Sigma_1) \# P$ regrouping  
$\cong \Sigma_{k+1} \# P$

Consider $\Pi_{2(k+1)} \cong \Pi_{2k+2} \cong \Pi_{2k} \# \Pi_2$  
$\cong (\Sigma_{k-1} \# K) \# (\Sigma_0 \# K)$ from our inductive assumption and case 2  
$\cong (\Sigma_{k-1} \# \Sigma_0) \# (K \# K)$ rearranging  
$\cong (\Sigma_{k-1}) \# (P \# P \# P \# P)$ since $K \cong P \# P$  
$\cong (\Sigma_{k-1}) \# (\Sigma_1 \# K)$ as seen in case 4  
$\cong (\Sigma_{k-1} \# \Sigma_1) \# K$ regrouping  
$\cong \Sigma_{(k+1)-1} \# K$

Thus, we have shown that every compact, non-orientable surface is homeomorphic to either $\Sigma_n \# P$ or $\Sigma_n \# K$, for some $n \geq 0$.

Which was to be proven.

$\textit{qed.}$
6. What are \( \pi_1(\text{cylinder}) \), and \( \pi_1(\text{Mobius strip}) \)? (Hint: Each is homotopy equivalent to something simpler.)

**Solution by Cynthia Flores.**

Consider \( f : \mathbb{S}^1 \times [-1, 1] \longrightarrow \mathbb{S}^1 \) s.t. \( f(x, s) = (x, 0) \) where \( x \in \mathbb{S}^1 \). I.e. \( f \) "squishes" the cylinder to the circle at the center of the cylinder.

Now consider \( g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \times [-1, 1] \) by \( g(x) \mapsto (x, 0) \). I.e. \( g(x) \) puts the circle in the center of the cylinder.

Well, we want \( g \circ f = f \) to be homotopic to the identity. Also, \( f \circ g = \text{id} \) (on the circle).

Let

\[
F : \text{cylinder} \times [0, 1] \longrightarrow \text{cylinder}
\]

\[
F((x, s), t) \mapsto (x, ts)
\]

When \( t = 0 \),
\[
F((x, s), 0) = (x, 0) = f(x, s).
\]

When \( t = 1 \),
\[
F((x, s), 1) = (x, s) = \text{id}.
\]

Therefore this is homotopy equivalence between cylinder and \( \mathbb{S}^1 \). Thus \( \pi_1(\mathbb{S}^1) = (\mathbb{Z}, +) \), this is the same for the cylinder.

Now, consider the Mobius strip. Since we are dealing with loops, clearly, all loops around the Mobius strip can be continuously deformed to the circle around the center of the strip. Similar to above, the Mobius strip is homotopy equivalent to \( \mathbb{S}^1 \).

Thus,

\[
\pi_1(\text{mobius strip}) = (\mathbb{Z},+)
\]

since the Mobius strip is homotopy equivalent to \( \mathbb{S}^1 \).
7. By considering fundamental groups, prove

(a) \( \mathbb{R}^2 - \{(0,0)\} \) is not homeomorphic to \( \mathbb{R}^2 \)

(b) \( \mathbb{R}^2 \) is not homeomorphic to \( \mathbb{R}^n \) for \( n \neq 2 \)

*Solution by Ashar Ali.*

Proof: First of all, we have that \( \pi_1(\mathbb{R}^2) \) is simply the trivial group because given any base point in \( \mathbb{R}^2 \), any closed loop that starts and ends at the base point can be continuously shrunk down to just that base point. Now let us consider \( \mathbb{R}^2 - \{(0,0)\} \). All of \( \mathbb{R}^2 - \{(0,0)\} \) can be continuously deformed as a homotopy equivalence to just \( S^1 \) by simply taking any vector in \( \mathbb{R}^2 - \{(0,0)\} \) and dividing by its norm. This way we will always end up on the unit circle around the origin. Notice that this is not a problem because our set doesn’t include \((0,0)\) so we never divide by zero. We already know that \( \pi_1(S^1) \) is \( \mathbb{Z} \), the set of all integers.

\[
\pi_1(S^1) \cong \mathbb{Z} \Rightarrow \pi_1(\mathbb{R}^2 - \{(0,0)\}) \cong \mathbb{Z};
\]

therefore

\[
\mathbb{R}^2 - \{(0,0)\} \not\cong \mathbb{R}^2
\]

because their fundamental groups differ.

(b) \( \mathbb{R}^2 \) is not homeomorphic to \( \mathbb{R}^n \) for \( n \neq 2 \).

Fact I: If \( A \cong B \), then \( A - \{p\} \cong B - \{f(p)\} \).

Therefore \( A - \{p\} \not\cong B - \{f(p)\} \Rightarrow A \not\cong B \).

Proof: Let us first consider the case \( n = 1 \). From part (a), we already have that \( \pi_1(\mathbb{R}^2 - \{(p)\}) \cong \mathbb{Z} \). But if we consider \( \mathbb{R} \), then \( \mathbb{R} - \{p\} \) where \( p \) is any real number, is not even connected. Therefore, \( \mathbb{R}^2 \not\cong \mathbb{R} \).

Now let us consider the case \( n > 2 \). First let us note that the fundamental group of \( \mathbb{R}^3 \) for example, is the trivial group but the fundamental group of \( \mathbb{R}^3 - \{p\} \) is also the trivial group. Because for any given base point, and for any closed loop based on that point, the loop can be continuously shrunk down to the base point itself. If the loop we are considering goes around the missing point, we can raise the loop in the third dimension and simply avoid the discontinuity. The same is true of \( \mathbb{R}^4, \mathbb{R}^5, \ldots, \mathbb{R}^n \). If a single point from \( \mathbb{R}^n \) for \( n > 2 \) is removed, \( \pi_1(\mathbb{R}^n - \{p\}) \cong \{1\} \) for \( n > 2 \). Therefore,

\[
\pi_1(\mathbb{R}^n - \{p\}) \cong \{1\} \not\cong \pi_1(\mathbb{R}^2 - \{p\}) \cong \mathbb{Z} \Rightarrow \mathbb{R}^n \not\cong \mathbb{R}^2 \text{ for } n > 2.
\]
8. Calculate the fundamental group of the space obtained by identifying the edges of a triangle according to the diagram below. (This space is not a surface, of course.)

\[ \pi_1(A) = \langle a \rangle, \quad \pi_1(A \cap B) = \langle \gamma \rangle, \quad \text{with } \gamma \text{ as the generator in } \pi_1(A \cap B). \]

\[ \gamma = a^2a^{-1} = a. \]

Therefore by Van Kampen’s theorem,
\[ \pi_1(D) = \langle a \rangle * \langle 1 \rangle = \langle a \rangle = \{1\}. \]

Solution by John Foss.

Let \( A \) be the punctured dunce hat and \( B \) be a disk with \( A \cap B \) a punctured disk. \( B \) is homotopy equivalent to a point, ergo \( \pi_1(B) = \{1\} \). To obtain \( \pi_1(A) \), homotopically project from the hole to the edges of the triangle. Gluing the edges together, we get \( a^2a^{-1} = a \), which is just the circle. Therefore \( \pi_1(A) = \langle a \rangle \). \( A \cap B \) is also homotopic to the circle, so \( \pi_1(A \cap B) = \langle \gamma \rangle \), with \( \gamma \) as the generator in \( \pi_1(A \cap B) \). Taking \( \gamma \) as an element of \( \pi_1(A) \) shows
\[ \gamma = a^2a^{-1} = a. \]

In \( \pi_1(B) \), \( \gamma = 1 \). Therefore by Van Kampen’s theorem,
\[ \pi_1(D) = \langle a \rangle * \langle 1 \rangle = \langle a \rangle = \{1\}. \]

The following picture shows how a typical loop is homotopic to a constant loop.
9. Compute the abelianization of $\pi_1(\Pi_n)$, for all $n$. For this, it is useful to use the fact from Problem 5 that $\Pi_{2k} \cong \Sigma_{k-1}\#K$ and $\Pi_{2k+1} \cong \Sigma_k\#P$.

Solution by Sarah Woolf.

Case 1:

Please see the picture.

$\Pi_{2k} \cong \Sigma_{k-1}\#K$.

$\Sigma_{k-1}\#K$ is a polygon with edges identified according to $a_1b_1\overline{a}_1\overline{b}_1 \ldots a_{k-1}b_{k-1}\overline{a}_{k-1}\overline{b}_{k-1}cde\overline{c}$.

Let $A = \text{an open boundary circle containing both points } x_0 \text{ and } x$.

Let $B = \Sigma_{k-1}\#K - \{x\}$.

$\pi_1(A, x_0) = 1$
$\pi_1(B, x_0) = \mathbb{Z} \ast \cdots \ast \mathbb{Z}, \text{ 2k times}$
$\pi_1(A \cap B, x_0) = \mathbb{Z} = \langle \delta \rangle$
$(i_A)_*(\delta) = 1$

$(i_B)_*(\delta) = a_1b_1\overline{a}_1\overline{b}_1 \ldots a_{k-1}b_{k-1}\overline{a}_{k-1}\overline{b}_{k-1}cde\overline{c}$

$\pi_1(\Sigma_{k-1}\#K, x_0) = \langle a_1, b_1, \ldots, a_{k-1}, b_{k-1}, c, d \mid a_1b_1\overline{a}_1\overline{b}_1 \ldots a_{k-1}b_{k-1}\overline{a}_{k-1}\overline{b}_{k-1}cde\overline{c} \rangle$.

The abelianization of this is: $\langle a_1, b_1, \ldots, a_{k-1}, b_{k-1}, c, d \mid \text{all commuting relations and } c^2 \rangle$.

This group is $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_2$, repeating 2k times.

Case 2:

$\Pi_{2k+1} \cong \Sigma_k\#P$.

$\Sigma_k\#P$ is a polygon with edges identified according to $a_1b_1\overline{a}_1\overline{b}_1 \ldots a_kb_k\overline{a}_k\overline{b}_k c^2$.

Let $A = \text{an open boundary circle containing both points } x_0 \text{ and } x$.

Let $B = \Sigma_{k-1}\#K - \{x\}$.

$\pi_1(A, x_0) = 1$
$\pi_1(B, x_0) = \mathbb{Z} \ast \cdots \ast \mathbb{Z}, \text{ 2k + 1 times}$
$\pi_1(A \cap B, x_0) = \mathbb{Z} = \langle \delta \rangle$

$(i_A)_*(\delta) = 1$

$(i_B)_*(\delta) = a_1b_1\overline{a}_1\overline{b}_1 \ldots a_kb_k\overline{a}_k\overline{b}_k c^2$

$\pi_1(\Sigma_k\#P, x_0) = \langle a_1, b_1, \ldots, a_k, b_k, c \mid a_1b_1\overline{a}_1\overline{b}_1 \ldots a_kb_k\overline{a}_k\overline{b}_k c^2 \rangle$.

The abelianization of this is: $\langle a_1, b_1, \ldots, a_k, b_k, c \mid \text{all commuting relations and } c^2 \rangle$.

This group is $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_2$, repeating 2k times.
10. Prove that the group with presentation \( \langle a, b \mid a^2b^2 \rangle \) is isomorphic to the group \( \langle x, y \mid xyx^{-1}y \rangle \).

11. Prove that the group with presentation \( \langle a, b, c \mid aba^{-1}b^{-1}c^2 \rangle \) is isomorphic to the group \( \langle x, y, z \mid x^2y^2z^2 \rangle \).

_Solutions by Cynthia Flores._

From lecture, given spaces \( A, B \) we know \( A \cong B \Rightarrow \pi_1(A) \cong \pi_1(B) \). Thus we will find two surfaces which are homeomorphic and compute their fundamental groups (expecting to find the group representations from above). Hence we will begin working backwards using Van Kampen’s Theorem.

Consider \( P#P = aabb \) where \( a, b \) label the identified edges of the polygon representation of \( P#P \).

Let \( A_1 \) be a open disk contained in the polygon containing \( x \) and \( x_0 \). Let \( B_1 = P#P - \{x_0\} \). Thus \( A_1 \cap B_1 = A_1 - \{x_0\} \).

Note: \( P#P = A_1 \cup B_1 \)

\[
\pi_1(A_1, x) = 1 \\
\pi_1(B_1, x) = \langle a, b \rangle
\]

We want \( \pi_1(A_1 \cup B_1, x) = \frac{\pi_1(A_1, x) \ast \pi_1(B_1, x)}{N} \) where \( N \) is the normal group or relations for all generators \( \gamma \in \pi_1(A_1 \cap B_1, x) \).

We will find our relations by the induced maps of \( \gamma \in \pi_1(A_1 \cap B_1, x) \)

\[
(i_{A_1})_*(\gamma) = 1 \\
(i_{B_1})_*(\gamma) = a^2b^2
\]

Thus,

\[
\pi_1(P#P) = \langle a, b \mid a^2b^2 \rangle
\]

Using the same approach on the Klein bottle, \( K = xyx^{-1}y \).

Let \( A_2 \subset K \) be an open disk containing \( x_0, \beta \). Let \( B_2 = K - x_0 \).

Again, \( K = A_2 \cup B_2 \)

\[
\pi_1(A_2, \beta) = 1 \\
\pi_1(B_2, \beta) = \langle x, y \rangle
\]

Again, we want \( \pi_1(A_2 \cup B_2, \beta) = \frac{\pi_1(A_2, \beta) \ast \pi_1(B_2, \beta)}{N} \) where \( N \) is the normal group or relations for all generators \( \gamma \in \pi_1(A_2 \cap B_2, \beta) \).

Well,

\[
A_2 \cap B_2 = A_2 - \{x_0\}
\]

We will find our relations by the induced maps of \( \gamma \in \pi_1(A_2 \cap B_2, \beta) \)
\[(i_{A_1})_*(\gamma) = 1\]
\[(i_{B_1})_*(\gamma) = xyx^{-1}y\]

Thus,
\[
\pi_1(K, \beta) = \langle x, y \mid xyx^{-1}y \rangle
\]

From lecture, we know \(K \cong P \# P\). Therefore, \(\langle x, y \mid xyx^{-1}y \rangle \cong \langle a, b \mid a^2b^2 \rangle\).

To prove that the group with presentation \(\langle a, b, c \mid aba^{-1}b^{-1}c^2 \rangle\) is isomorphic to the group \(\langle x, y, z \mid x^2y^2z^2 \rangle\), we will use the result from February 12 (4): \(T^2\# P\) is homeomorphic to \(K\# P\).

\(T^2\# P\) is the surface obtained by identifying the edges of a hexagon according to the pattern \(aba^{-1}b^{-1}c^2\). Similar as before, \(\pi_1(T^2\# P) = \langle a, b, c \mid aba^{-1}b^{-1}c^2 \rangle\).

\(K\# P\) is the surface obtained by identifying edges of a hexagon according to the pattern \(x^2y^2z^2\). Again, we find \(\pi_1(K\# P) = \langle x, y, z \mid x^2y^2z^2 \rangle\).

Since \(T^2\# P\) is homeomorphic to \(K\# P\) this implies
\[
\pi_1(T^2\# P) \cong \pi_1(K\# P)
\]
\[
\Rightarrow \langle a, b, c \mid aba^{-1}b^{-1}c^2 \rangle \cong \langle x, y, z \mid x^2y^2z^2 \rangle.
\]
12. Construct an example of a space whose fundamental group is $\mathbb{Z}_n$. (This does not have to be a surface; in fact, it cannot be one.)

*Solution by Elizabeth Leyton.*

*Proof.* Consider a polygon with $n$ identified edges which has the representation:

$$P = a^n$$

Let $A$ be an open disc removed from this polygon and $B$ be the polygon minus one point. Then $A \cap B$ is an open disc removed from the polygon minus one point. Then we have:

$$\pi_1(A) = 1$$

$$\pi_1(B) = \mathbb{Z}$$

and

$$\pi_1(A \cap B) = \mathbb{Z}$$

Let $\gamma \in \pi_1(A \cap B)$ be a generator. Then $i_A(\gamma)$ is a loop in $A$. This means that $\gamma$ is homotopic to a single point. Therefore,

$$i_A(\gamma) = 1$$

Similarly, $i_B(\gamma)$ is a loop in $B$. This means that $\gamma$ is homotopic to the perimeter of the polygon, which is simply $a^n$. Therefore,

$$i_B(\gamma) = a^n$$

This yields the relation:

$$a^n = 1$$

Therefore, by Van Kampen’s theorem, this gives:

$$\pi_1(P) = \langle a | a^n \rangle = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$$
13. Represent the following triangulated surface as a polygon with identified edges, and identify the surface by computing the abelianization of its fundamental group.

\[
\begin{array}{cccccc}
123 & 234 & 345 & 451 \\
512 & 136 & 246 & 356 \\
416 & 526 \\
\end{array}
\]

*Solution by Stacey Disbrow.*

**Solution:** Consider this representation of the surface with the given triangulation:

Removing the triangulation and simply looking at the perimeter polygon with identified edges yields:

Let:

1 → 2 = a
2 → 6 = b
1 → 6 = c

Then we have:

where \( c' = c^{-1} = \bar{c} \).
Reading counterclockwise, we have $ab\overline{c}ab\overline{c}$.

However, since this was just a triangulation, we may simplify.

Since $ab\overline{c}$ is repeated, let $d = ab\overline{c}$.

Then our polygon becomes:

![Diagram](image1)

From prior knowledge, this surface is obviously the Projective Plane, but let us verify this.

Let us call our polygon $P$.

Let $A$ be the open disk centered at point $x$, and including the point $x_0$.

![Diagram](image2)

Then $\pi_1(A, x_0) = 1$

Let $B = P - \{x\}$, our polygon without the center point.

![Diagram](image3)

Then $\pi_1(B, x_0) = \mathbb{Z}$, simply the integers.

Now, $A \cap B$ is the open disk without the center point.

![Diagram](image4)

$\pi_1(A \cap B) = \mathbb{Z}$, also the integers.

Now, let us look at our relations.

Taking a loop $\gamma \in A \cap B$ starting at $x_0$ and looking at it as a loop in $A$, we have:

$\langle i_A \rangle \cdot (\gamma) = \{1\}$
Looking at $\gamma$ as a loop in $B$, we have:

$$(i_B)_*(\gamma) = d^2$$

By Van Kampen’s Theorem, $\pi_1(P, x_0) \cong \pi_1(A \cup B, x_0) \cong \frac{1 \times <d>}{N} \cong \frac{<d>}{d^2}$

Normally, one would compute the abelianization of this surface’s fundamental group, but in this case there is nothing to abelianize. This is already an abelian group.

So, we have $\pi_1(P, x_0) \cong <d | d^2 \cong Z_2$, the set of congruence classes modulo 2.

This is known to be the fundamental group of the Projective Plane.

Thus we have verified that our unknown surface is the Projective Plane.