Lefschetz Fibrations of 4-Dimensional Manifolds

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0   Introduction

Since the 1980s, the subject of 4-dimensional manifold topology has experienced dramatic growth. Accompanying this growth has been a number of surprising, even shocking, results which demonstrate rich and complicated relationships between different categories of manifold, relationships which are unique to dimension four.

An n-dimensional manifold is an object which locally resembles n-dimensional Euclidean space. Different categories of manifolds can be considered simply by requiring different sorts of maps to perform these local identifications: A manifold may be *smooth* (if the maps are required to be infinitely differentiable), or *complex* (if n is even and the maps are required to be holomorphic), or *topological* (if the maps are merely required to be continuous), or many things in between.

Prior to 1980, smooth 4-manifold theory consisted largely of describing examples of complex manifolds as handlebodies, and trying to manipulate these handlebodies using the so-called *Kirby calculus* in order to construct interesting new examples, or to show that

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known examples were diffeomorphic. In short, 4-manifold topology was the province of geometric topologists. It was conjectured, based on little evidence other than the lack of examples to the contrary, that every simply connected smooth 4-manifold could be expressed as a connected sum of complex 4-manifolds, where the complex summands were allowed to have either one of its orientations. (The sphere S^4, known to be not complex, was excluded from this conjecture.) In 1980, however, came a revolution in the subject, as Simon Donaldson introduced powerful ideas from gauge theory and mathematical physics into the field, which succeeded in establishing many significant negative results, often showing that smooth 4-manifolds with prescribed properties could not exist, or that two known examples were not diffeomorphic. Donaldson theory, a major mathematical industry in the 80s and early 90s, was instrumental in establishing the following theorem of Robert Gompf and Tom Mrowka, which showed that complex manifolds alone were not sufficient building blocks for all smooth 4-manifolds. We say that a 4-manifold is *irreducible* if it is not a connected sum of nontrivial pieces.

Theorem. *([GM]) There exist infinite families of simply connected irreducible smooth 4-manifolds which are not complex.*

Another category of manifold, originating in mathematical physics, is a *symplectic* manifold. (A definition is given below.) Defined only in even dimensions, a symplectic manifold is a smooth manifold with admits a closed nondegenerate 2-form of the sort found natively on a large class of complex manifolds (including all simply connected ones). Hence symplectic 4-manifolds can be thought of as a smooth generalization of complex 4-manifolds.

The mid-1990s saw a number of developments which made symplectic 4-manifolds natural candidates to be the building blocks of all smooth 4-manifolds. Gompf showed that a basic cut-and-paste operation, a symplectic normal sum, could be performed on symplectic manifolds, with the result being symplectic [G1]. Cliff Taubes showed that the newly discovered Seiberg-Witten theory–an extension of Donaldson theory– could be used to produce invariants for symplectic 4-manifolds [T], a necessary step for showing examples to be irreducible. Combining these breakthroughs, Zoltan Szabo proved that the symplectic category was not a sufficient source of irreducible 4-manifolds.

Theorem. *([Sz]) There exist simply connected irreducible smooth 4-manifolds which are not symplectic.*

Szabo's example was soon generalized by Ron Fintushel and Ron Stern, who showed that one can find infinite families of simply connected irreducible non-symplectic smooth 4-manifolds [FS1].

We thus have a nesting of categories

$$\{\text{Complex}\} \subset \{\text{Symplectic}\} \subset \{\text{Smooth}\}$$

for simply connected 4-manifolds, together with an abundance of irreducible examples showing these to be distinct. In an effort to sort out the poorly understood relationship between these categories, a natural question arises: Is there a purely topological description of symplectic 4-manifolds? An answer was provided with companion theorems of Donaldson and Gompf: roughly speaking, symplectic 4-manifolds are those that (after perhaps blowing up) admit the structure of a *Lefschetz fibration*. A Lefschetz fibration is a fibering of a 4-manifold by surfaces, with a finite number of the fibers permitted to have singularities of a prescribed type. These fibrations were first discovered on complex surfaces by Lefschetz, who used them as a tool in the study of their topology. Thus symplectic 4-manifolds may be viewed topologically as those admitting a fibration structure resembling that found on complex surfaces.

In this article, we survey some recent results on symplectic Lefschetz fibrations. We note at the outset that a definitive introduction to modern smooth 4-manifold theory from a topological viewpoint, Gompf and Andras Stipsicz's "4-Manifolds and Kirby Calculus" [GS], has recently been written. Their textbook includes sections on many aspects of 4-manifold topology, and features a detailed chapter on Lefschetz fibrations. This article is meant as a kind of appendix to Chapter 8 of [GS], providing elaboration on some of the results mentioned in their book, and giving an update on some more recent results.

1   Definitions

We begin with an official definition of a manifold.

Definitions. A separable Hausdorff topological space X is a *topological n-dimensional manifold* or *topological n-manifold* if for every point $p \in X$ there is an open neighborhood U of p in X and a homeomorphism $\phi : U \to \mathbf{R}^n$. The pair (U, ϕ) is called a *chart*. The space X is, in addition, a *smooth n-manifold* if given any two charts with open sets U_α and U_β and homeomorphisms $\phi_\alpha : U_\alpha \to \mathbf{R}^n$ and $\phi_\beta : U_\beta \to \mathbf{R}^n$, respectively, then the transition functions

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

between open subsets of $\mathbf{R}^n$ are infinitely differentiable. If X is a smooth $2n$-manifold, then we may identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$ in the usual way, in which case X is *complex* if the transition functions $\phi_\beta \circ \phi_\alpha^{-1}$ are holomorphic.

There may be many incompatible ways to assign charts to a topological space making it into a smooth or complex manifold. Any particular selection of charts (up to a natural equivalence) is called a smooth or complex *structure*. One may also consider smooth *manifolds with boundary* by allowing the homeomorphisms in the charts to have range either

$\mathbf{R}^n$ or $\mathbf{R}_+^n = \{(x_1, \ldots, x_n) : x_n \geq 0\}$. The analogous notion of a complex manifold "with boundary" is called a *Stein manifold*, which we discuss in Section 7. We will use the phrase "complex surface" to refer to a complex manifold with two complex dimensions, which will be a real 4-manifold. The unadorned phrase "surface" with refer to a real 2-manifold.

In this article, we will be primarily concerned with 4-manifolds which are simply connected (i.e. those with a trivial fundamental group). Simply connected complex surfaces fall into the well-studied class of *Kahler* surfaces, which are known to admit a closed nondegenerate 2-form. Generalizing this property to the smooth category gives the following.

Definition. A smooth 4-manifold X is *symplectic* if it admits a closed nondegenerate 2-form $\omega \in \Omega^2(X, \mathbf{R})$.

2   Lefschetz Fibrations and Symplectic 4-manifolds

We now give a definition for the central topic of this paper.

Definition. Let X be a compact, connected, oriented, smooth 4-manifold. A *Lefschetz fibration* on X is a map $f : X \to C$, where C is a compact, oriented, smooth 2-manifold, such that each critical point of f has an orientation-preserving chart on which $f : \mathbf{C}^2 \to \mathbf{C}$ is given by $f(w, z) = wz$.

It is a consequence of Sard's Theorem that pre-images $f^{-1}(x)$ are diffeomorphic to a compact 2-manifold Σ_g of a fixed genus g, as long as x is not one of the finitely many critical values of f. We may assume that each of the critical points of f lies in a different fiber of f. Since $f(w, z) = wz$ in a neighborhood of a critical point, we see that the singular fiber (corresponding to $f^{-1}(0)$ locally) is an immersed surface with a single transverse self-intersection. For intuition, therefore, a Lefschetz fibration should be pictured as a smooth fibration of X by surfaces Σ_g, with finitely many singular fibers, each of which has a single transverse self-intersection. We will often refer to a Lefschetz fibration, according to its fiber genus, as a *genus g Lefschetz fibration*. If a Lefschetz fibration has the property that no singular fiber contains an embedded sphere of self- intersection -1, it is termed a *relatively minimal* Lefschetz fibration. Since any sphere of this sort can be blown down in a way which preserves the fibration, this condition can always be arranged, and some authors incorporate it into their definition of a Lefschetz fibration. Except where stated otherwise, we will assume all Lefschetz fibrations are relatively minimal.

If X is a Kahler complex surface, then X is known to admit a holomorphic Lefschetz pencil of curves, which can be blown up to yield a holomorphic Lefschetz fibration $X \# n\overline{\mathbf{CP}^2} \to \mathbf{CP}^1$. (This construction is described in detail from a topological perspective
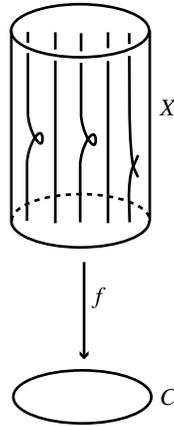
Figure 1: Lefschetz fibration

in Section 8.1 of [GS].) Our definition of a Lefschetz fibration is therefore a generalization of this construction to the smooth category. In this generalization, we allow bases other than $\mathbf{CP}^1 = S^2$. In addition, our definition allows manifolds with boundary, with the singular fibers necessarily in the interior of X. The total space X will be simply connected only when $C = S^2$ or $C = D^2$, and we will primarily consider those cases here.

The fact that Lefschetz fibrations characterize symplectic 4-manifolds follows from deep theorems of Donaldson and Gompf.

Theorems. *(a.) ([D]) For any symplectic 4-manifold X, there exists a nonnegative integer n such that the n-fold blowup $X \# n\overline{\mathbf{CP}^2}$ of X admits a Lefschetz fibration $f : X \# n\overline{\mathbf{CP}^2} \to S^2$.*

(b.) ([GS]) If a 4-manifold X admits a genus g Lefschetz fibration $f : X \to C$ with $g \geq 2$, then it has a symplectic structure.

Thus Lefschetz fibrations provide a topological way to study symplectic 4-manifolds. The restriction that $g \geq 2$ in (b.) is mild, and rules out only a collection of well-understood examples (certain torus-bundles over tori and their blowups are not symplectic, yet admit genus 1 fibrations).

The assumption of orientation-preserving charts in the definition of Lefschetz pencils and fibrations is a subtle but crucial point. If the definitions are relaxed to allow orientation-reversing charts as well, X can no longer be shown to be symplectic. These broader constructions are known as *achiral* Lefschetz pencils and fibrations; lacking a connection to symplectic structures, we will not consider them here.

3   The Topology of Lefschetz Fibrations

We begin by describing the topology of a Lefschetz fibration locally, in the neighborhood of one singular fiber. Let $f : X \to D^2$ denote a Lefschetz fibration with only one singular fiber, say $F_1 = f^{-1}(x_1)$, in the interior of D^2. Additionally, let $F_0 = f^{-1}(x_0)$ be a nearby regular (i.e. nonsingular) fiber in the interior of D^2, and assume that we have made an explicit identification of the fiber F_0 with a standard genus g surface Σ_g, as in Figure 2. We can visualize the singular fiber F_1 as being obtained by taking a simple closed curve γ in F_0 and gradually shrinking it to a point as we approach F_1. The curve γ which describes the
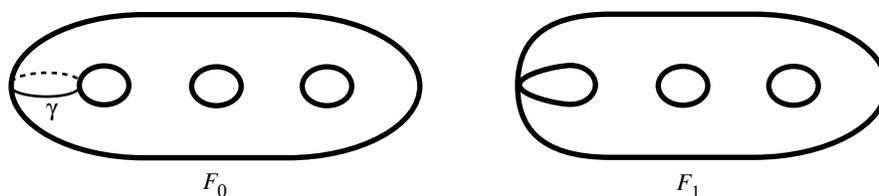

Figure 2: The vanishing cycle γ

singular fiber is called the *vanishing cycle* for that fiber. Furthermore, X can be described concretely as a handlebody obtained by adding a 2-handle to $\Sigma_g \times D^2$ with attaching circle γ. Intuitively, this handlebody description makes sense: In the absence of any singular fibers, X is just $\Sigma_g \times D^2$. Attaching a 2-handle along γ provides the necessary disk (the core of the handle) to shrink γ to a point. To preserve the fibering, a delicate framing condition must be satisfied: The 2-handle must be attached with framing -1 relative to the product framing on $\partial(\Sigma_g \times D^2) = \Sigma_g \times S^1$. The Morse theoretic arguments justifying this handlebody description and the framing can be found in [Ka].

The vanishing cycle γ completely determines the topology of a neighborhood of a singular fiber, up to diffeomorphism. Indeed, it is not hard to see that the neighborhoods of singular fibers given by any two nonseparating curves will be diffeomorphic. This follows from the fact that given any two nonseparating simple closed curves representing vanishing cycles, there is a diffeomorphism of Σ_g taking one to the other, hence one can easily map the handlebody descriptions of the corresponding Lefschetz fibrations to one another. A similar statement is true for separating curves, as long as they separate Σ_g into surfaces of the same genus. Hence neighborhoods of singular fibers in genus g Lefschetz fibrations can be classified, up to diffeomorphism: There are $1 + \left[\frac{g}{2}\right]$ of them, one given by those with a nonseparating vanishing cycle, and the others given by those with a separating vanishing cycle which separates off a surface of genus h, with $1 \le h \le \left[\frac{g}{2}\right]$.[2]

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[2]The reader may wonder about nullhomotopic vanishing cycles, which separate Σ_g into surfaces of genus

nonseparating vanishing cycle   separating vanishing cycle
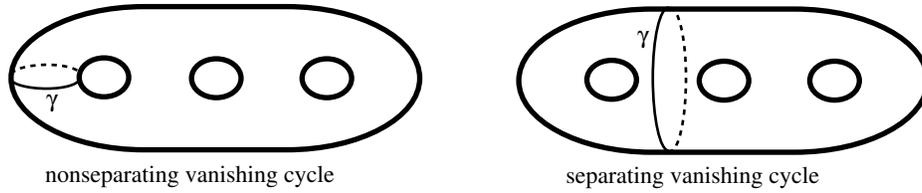
Figure 3:

The vanishing cycles also play a crucial role in understanding the boundary of a neigh-borhood of a singular fiber. Since we assumed the singular fiber of f to be in the interior of X, all of the fibers lying above the boundary circle S^1 of the base are nonsingular. In other words, the boundary ∂X is a Σ_g-bundle over S^1. As a result we can describe it as

$$\partial X = \frac{\Sigma_g \times I}{((\psi(x), 0) \sim (x, 1)},$$

where $\psi : \Sigma_g \to \Sigma_g$ is a homeomorphism. The map ψ is called the *monodromy* of the singular fiber; as with the vanishing cycle, it depends on our choice of an identification of a regular fiber F_0 with a genus g surface Σ_g. Intuitively, the monodromy documents how a fiber changes if we traverse the boundary once along its S^1 factor. In this case, where the Σ_g-bundle over S^1 arises as the boundary of a neighborhood of a singular fiber in a Lefschetz fibration, this monodromy has a useful description: It is given by a right-handed Dehn twist about the vanishing cycle γ for that fiber. A Dehn twist $D_\gamma : \Sigma_g \to \Sigma_g$ is the homeomorphism given by removing a cylindrical neighborhood of γ on Σ_g, and regluing it after giving a full 360° twist about one end, as in Figure 4. Since we may twist in
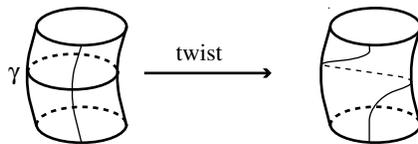

Figure 4: The Dehn twist D_γ

either direction, both right- and left-handed Dehn twists make sense. The fact that only right-handed Dehn twists arise as monodromies of Lefschetz fibrations is a consequence of requiring orientation-preserving charts in the definition of a Lefschetz fibration; considering achiral Lefschetz fibrations would allow both kinds of Dehn twists.

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0 and g. In this case, the resulting singular fiber will have a spherical component of square -1, hence the resulting Lefschetz fibration will not be relatively minimal.

Since the local topology of each singular fiber in a Lefschetz fibration is understood, the challenge in studying Lefschetz fibrations becomes knowing how these local models can fit together when there are many singular fibers. Assume now that $f : X \to D^2$ is a Lefschetz fibration with singular fibers $F_1 = f^{-1}(x_1), \ldots, F_\mu = f^{-1}(x_\mu)$ in the interior of X. If we
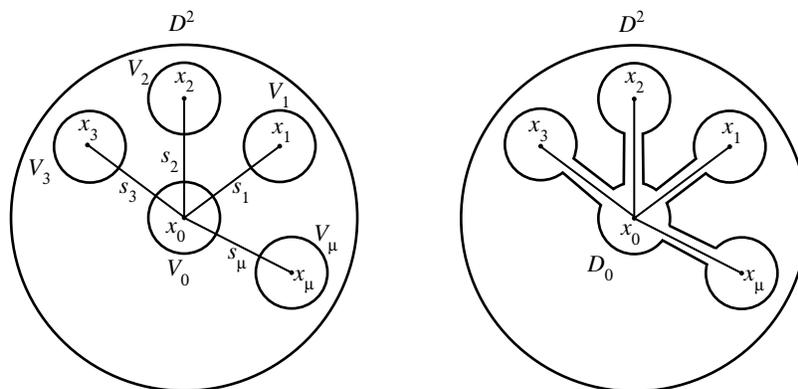

Figure 5:

pick a collection of small disjoint disks $V_1, \ldots, V_\mu$ with each $x_i \in V_i$, then f restricted to each $f^{-1}(V_i)$ is a Lefschetz fibration over D^2 with only one singular fiber, and so its topology (i.e. monodromy, diffeomorphism type) is encoded by a vanishing cycle γ_i in a nearby nonsingular fiber, as described above. We can relate the descriptions of these different singular fibers as follows. Let $F_0 = f^{-1}(x_0)$ denote a fixed nonsingular fiber, and select a collection of arcs $s_1, \ldots, s_\mu$ going from x_0 to each x_i, respectively. We may assume that the arcs $s_1, \ldots, s_\mu$ are indexed so that they appear in order as we move counterclockwise about x_0. Each $f^{-1}(s_i)$ gives a trivial Σ_g-bundle over I along which we can transport the identification of the common reference fiber F_0 with Σ_g to the nearby nonsingular fiber in V_i carrying the vanishing cycle. Each singular fiber F_i can then be described by a vanishing cycle γ_i with respect to a common identification with Σ_g. If we let

$$D_0 = V_0 \bigcup_{i=1}^{\mu} (\text{neighborhood}(s_i) \cup V_i)$$

(see Figure 5), then since D_0 is a large disk containing all of the singular values of f, $f^{-1}(D_0) \subset X$ is diffeomorphic to X (their difference is a trivial collar neighborhood of ∂X). This allows us to describe X as $\Sigma_g \times D^2 \bigcup_{i=1}^{\mu} H_i$, where each H_i is a 2-handle attached along the vanishing cycle γ_i, subject to the same framing condition mentioned above, with the handles attached in order in distinct Σ_g fibers. This argument also shows that the monodromy about ∂D^2 is given by the composition of Dehn twists $D_{\gamma_1} D_{\gamma_2} \cdots D_{\gamma_\mu}$. (We

will write compositions of Dehn twists in monodromies as words, from left-to-right.) This composition is called the *global monodromy* of f. Let M_g denote the *mapping class group* of Σ_g, the group of self-homeomorphisms of Σ_g, modulo isotopy. The global monodromy of f is typically regarded as an element of M_g given by the composition of right-handed Dehn twists about the vanishing cycles of f.

Thus the topology of a Lefschetz fibration over D^2 seems completely determined by the ordered collection $(\gamma_1, \ldots, \gamma_\mu)$ of vanishing cycles. This is not quite true: The list of curves depends implicitly on choices made during the above description, and other choices yield equivalent Lefschetz fibrations. For one thing, we may cyclically permute the list $(\gamma_1, \ldots, \gamma_\mu)$ and arrive at the same fibration. Note, too, that we may use a different identification of the central fiber F_0 with Σ_g, which will have the effect of conjugating all of the Dehn twists D_{γ_i} in the global monodromy by a fixed element $\psi \in M_g$. (Since $\psi \circ D_{\gamma_i} \circ \psi^{-1} = D_{\psi(\gamma_i)}$, the result will still be a Dehn twist, just about a different vanishing cycle.) Furthermore, different choices of arcs s_i will give a different description of the vanishing cycles, and therefore of the Dehn twists comprising the monodromy. For instance, changing the arcs as in Figure 6 will change the vanishing cycles from $(\cdots, \gamma_i, \gamma_{i+1}, \cdots)$ to $(\cdots, \gamma_{i+1}, D_{\gamma_{i+1}}(\gamma_i), \cdots)$; this move and similar ones are known as *elementary transformations*. Although elementary
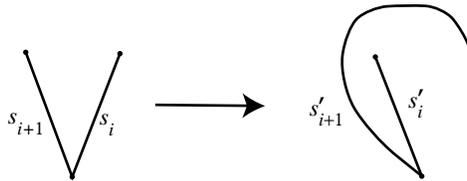

Figure 6: Elementary transformations

transformations alter the description of the monodromy about individual singular fibers, the global monodromy is unaffected. Two Lefschetz fibrations are equivalent if and only if it is possible to transform the monodromy of one (expressed in terms of Dehn twists) into the other through a combination of elementary transformations (and their inverses), and conjugation by elements of M_g.[3]

Consider now Lefschetz fibrations f over S^2. We may split S^2 into two hemispheres $D^2 \cup D^2$ so that one of the D^2's contains all of the singular values of f, in which case

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[3]It is possible to condense the monodromy about different singular fibers into a single monodromy representation $\pi_1(D^2 - \{x_1, \ldots, x_\mu\}, x_0) \to M_g$. The collection of arcs s_i describe loops in $D^2 - \{x_1, \ldots, x_\mu\}$ based at x_0 given by following s_i to V_i, traversing ∂V_i once counterclockwise, and returning to x_0 along s_i. These loops form a basis of $\pi_1(D^2 - \{x_1, \ldots, x_\mu\}, x_0)$, and the elementary transformations ensure that this representation is independent of a choice of basis.

the global monodromy over that hemisphere can be expressed in terms of a collection of vanishing cycles as $D_{\gamma_1} D_{\gamma_2} \cdots D_{\gamma_\mu}$. However, this fibration on the boundary must be the trivial product $\Sigma_g \times S^1$, in order to extend over the other hemisphere as the trivial fibration $\Sigma_g \times D^2$. This means that $D_{\gamma_1} D_{\gamma_2} \cdots D_{\gamma_\mu}$ must be isotopic to the identity. Conversely, a composition $D_{\gamma_1} D_{\gamma_2} \cdots D_{\gamma_\mu}$ of Dehn twists determines a Lefschetz fibration over D^2, and if this composition is isotopic to the identity we can extend the fibration (uniquely for $g \geq 2$) to get a Lefschetz fibration over D^2. This provides a purely group theoretic way to describe Lefschetz fibrations over S^2, and in light of the theorems of Donaldson and Gompf, a combinatorial classification of symplectic 4-manifolds (up to blowing up). This is summarized in the next Proposition.

Proposition. *For any fixed $g \geq 2$, there is a one-to-one correspondence between genus g Lefschetz fibrations over S^2 and relations of the form $D_{\gamma_1} D_{\gamma_2} \cdots D_{\gamma_\mu}$ in M_g, modulo elementary transformations and the action of M_g by conjugation.*

Given two Lefschetz fibrations of the same fiber genus, we can combine them into another Lefschetz fibration.

Definition. Let $X_1 \to C_1$ and $X_2 \to C_2$ be two genus g Lefschetz fibrations, and let $F_1 \subset X_1$ and $F_2 \subset X_2$ be two regular fibers. We identify neighborhoods of each F_i with $F_i \times D^2$, and select a diffeomorphism $h : F_1 \to F_2$. The *fiber sum* $X_1 \#_F X_2$ is defined as the manifold $(X_1 - F_1 \times D^2) \cup_\psi (X_2 - F_2 \times D^2)$, where $\psi : \partial(F_1 \times D^2) \to \partial(F_2 \times D^2)$ is given by $h \times$ (complex conjugation) $: F_1 \times S^1 \to F_2 \times S^1$.

The fiber sum construction yields a Lefschetz fibration $X_1 \#_F X_2 \to C_1 \# C_2$. This fibration and the diffeomorphism type of $X_1 \#_F X_2$ depend on the choice of the identification h of regular fibers.

4   Examples

The monodromy classification gives a way to list examples of Lefschetz fibrations, in a way that is complete for genus $g = 1$, and partly so for $g = 2$. Unfortunately, known presentations of mapping class groups for $g \geq 3$ tend to have relations which feature both left- and right-handed Dehn twists, and as a result they provide less direct information about monodromies of symplectic Lefschetz fibrations. The examples we give below are standard, and are discussed in [GS]. In Section 6, we elaborate on these examples, describing them more fully from a branched covering perspective.

Elliptic (g = 1) examples. Let α_1 and α_2 be the curves on the torus Σ_1 pictured in Figure 7, and let a_1 and a_2 denote Dehn twists about α_1 and α_2, respectively. The mapping
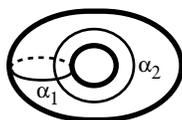
Figure 7: Σ_1

class group M_1 is well-known to have a presentation with generators a_1 and a_2, and relation $(a_1 a_2)^6 = 1$. This relation defines an elliptic fibration over D^2, which can be extended to an elliptic fibration $E(1) \to S^2$.[4] We can then form the n-fold fiber sum (using the identity homeomorphism on regular fibers) $E(n) = \#_F(nE(1))$, resulting in an example with global monodromy $(a_1 a_2)^{6n}$. It was proven by Moishezon that the global monodromy of any elliptic Lefschetz fibration is equivalent to this relation [Mo], hence the family of $E(n)$'s are a complete classification of genus 1 Lefschetz fibrations with at least one singular fiber. Each $E(n)$ is complex.

Higher genus examples. Let $\alpha_1, \alpha_2, \ldots, \alpha_{2g+1}$ be the curves indicated on Figure 8, and let a_i denote a right-handed Dehn twist D_{α_i} about α_i. Then the following equations hold
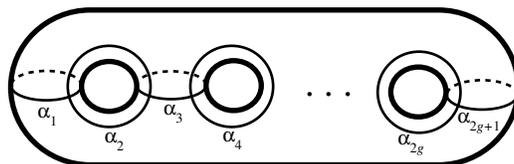

Figure 8: Σ_g

in M_g:

$$(a_1 a_2 \cdots a_{2g+1}^2 \cdots a_2 a_1)^2 = 1 \tag{1}$$

$$(a_1 a_2 \cdots a_{2g})^{2(2g+1)} = 1 \tag{2}$$

$$(a_1 a_2 \cdots a_{2g+1})^{2g+2} = 1. \tag{3}$$

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[4]When $g = 1$, possible extensions are not unique, and must be chosen with care. The boundary of the total space of the fibration over D^2 is diffeomorphic to $T^2 \times S^1 = S^1 \times S^1 \times S^1$, which admits many nontrivial diffeomorphisms which can be used to attach $T^2 \times D^2$ and produce a closed 4-manifold fibered by tori. However, only those attached by maps of the form $\phi \times$ id will produce Lefschetz fibrations; other choices of attaching maps will produce non-Lefschetz fibrations with more complicated singular fibers known as *multiple fibers*.

This gives rise to Lefschetz fibrations given by equations (1)-(3), with total spaces $X(1), X(2)$, and $X(3)$, respectively. These examples are complex, and for $g = 2$ it was shown by Kenneth Chakiris that any holomorphic Lefschetz fibration with only nonseparating vanishing cycles is a fiber sum of one of these three [Ch]. (A simpler proof has been discovered by Ivan Smith [Sm2].)

We also give a useful example of a Lefschetz fibration with some of the vanishing cycles given by separating curves. This example was discovered by Yukio Matsumoto for genus $g = 2$ [Ma], and has been extended to arbitrary genus by Carlos Cadavid [Ca]. (See also [Ko].) Let $\beta_1, \beta_2, \beta_3, \beta_4$ be the curves on Σ_2 indicated in Figure 9, and let b_i denote a right-handed Dehn twist D_{β_i}. Then the equation

$$(b_1 b_2 b_3 b_4)^2 = 1 \tag{4}$$

holds in M_2. Let $X(4)$ denote the Lefschetz fibration over S^2 obtained from the relation
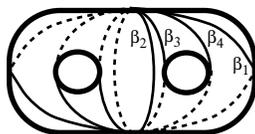

Figure 9: Σ_2

(4). This fibration is complex, although as we will see below, it can be usefully exploited to construct noncomplex fibrations.

5   Complex versus Symplectic Lefschetz Fibrations

As mentioned in the Introduction, there are now many known examples of noncomplex symplectic 4-manifolds. Donaldson's theorem guarantees that they admit a Lefschetz fibration (after perhaps being blown up), but is non-constructive. Explicit constructions of noncomplex symplectic Lefschetz fibrations have been given by several authors, all of whom form fiber sums of known complex fibrations using a nontrivial diffeomorphism of a regular fiber. Non-simply connected examples were given by independently by Smith [Sm1], and by Burak Ozbagci and Stipsicz [OS], while simply connected examples were discovered by Fintushel and Stern [FS2] (although they did not explicitly determine the vanishing cycle structure of their examples). We discuss the examples of Ozbagci and Stipsicz (Smith's are similar).

Theorem. *([OS]) There are infinitely many (pairwise nonhomeomorphic) 4-manifolds which admit genus 2 Lefschetz fibrations but which are not complex (with either orientation).*

Their construction is to form the fiber sum B_n of two copies of the example $X(4)$ above, using the diffeomorphism h^n, where h is a Dehn twist about the curve α_5 in Figure 8 (with $g = 2$). They establish their theorem by calculating $\pi_1(B_n) = \mathbf{Z} \oplus \mathbf{Z}_n$ (thereby distinguishing B_n for different values of n), and by consulting the Kodaira classification of complex surfaces to show that the n-fold cover M_n of B_n cannot be complex.

Thus, noncomplex Lefschetz fibrations can be formed by taking fiber sums of complex ones, raising the question of whether every Lefschetz fibration was at the very least a fiber sum of complex ones. A negative answer was given independently by Smith and Stipsicz.

Theorem. *([Sm3]; [St1]) There exist infinitely many simply connected non-complex Lefschetz fibrations which do not decompose as non-trivial fiber sums.*

Both Smith and Stipsicz give a nonconstructive proof based on a Lemma asserting that if a Lefschetz fibration $f : X \to S^2$ admits a section $g : S^2 \to X$ with $f \circ g = $ identity and image a sphere of self-intersection number -1, then it cannot decompose as a fiber sum. (Smith gives a clever elementary proof of this Lemma using hyperbolic geometry; Stipsicz proves it using results from Seiberg-Witten theory.) Since any Lefschetz fibration obtained by blowing up a Lefschetz pencil has such sections, examples are plentiful.

In addition, the first explicit example of an indecomposable (into fiber sums) noncomplex Lefschetz fibration has recently been given by Smith [Sm4]. In the mapping class group M_3, the relation

$$dea_4a_3a_2a_1a_5a_4a_3a_2a_6a_5a_4a_3(a_1a_2a_3a_4a_5a_6)^{10} = 1 \qquad (5)$$

holds, where $d = D_\delta, e = D_\varepsilon$, and $a_i = D_{\alpha_i}$ for the curves pictured in Figure 10. (This
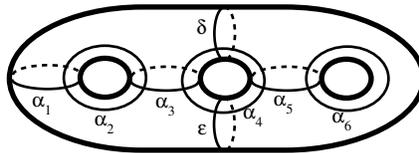

Figure 10:

relation was first derived by the author.) Using calculations of intersection numbers between a collection of associated spheres in the moduli space of curves and certain natural divisors, Smith shows that the total space of the Lefschetz fibration given by (5) cannot be complex.

6   Lefschetz Fibrations and Branched Covers

One way that Lefschetz fibrations have been successfully studied is through their close relationship to the construction of a branched cover.

Definition. A smooth map $\pi : M^m \to N^m$ between manifolds of the same dimension is called an *n-fold branched cover* with *branch set* $B \subset N$ if $\pi|M - \pi^{-1}(B) : M - \pi^{-1}(B) \to N - B$ is an n-fold covering space, and if for each $b \in B$ there are charts on which $\pi : \mathbf{C} \times \mathbf{R}_+^{m-2} \to \mathbf{C} \times \mathbf{R}_+^{m-2}$ is given by $\pi(z, x) = (z^k, x)$, for some positive integer k. The branched cover is called *regular* if the associated (unbranched) cover $\pi|M - \pi^{-1}(B)$ is regular. If $\pi^{-1}(b)$ consists of $n - 1$ points for all $b \in B$, then π is called *simple*.

Branched covers can be used to construct fibrations, as follows. Let Y be either an S^2-bundle over S^2 or the connected sum $\mathbf{CP}^2 \# k\overline{\mathbf{CP}^2}$, so that there is a projection $p : Y \to S^2$ whose fibers are spheres.[5] Let $\pi : X \to Y$ be a branched cover with branch set $B \subset Y$. If the branch set B is suitably transverse to the fibers of p, then the composition $\pi \circ p$ will be a Lefschetz fibration, one which we will say is *obtained from the branched cover* π. In particular, this means that the Lefschetz fibration restricted to each fiber is a branched cover of surfaces; this is true even for the singular fibers.

Proposition. *Each of the examples of Lefschetz fibrations $X(1), X(2),$ and $X(3)$ in Section 4 can be obtained as a 2-fold branched cover π of an S^2-bundle over S^2, branched over an embedded surface.*

The proof of this proposition appears essentially in [F1], where the examples $X(1)$ and $X(2)$ are discussed in detail for $g = 2$. The argument generalizes in a straightforward way to cover all g (see the diagrams in [GS]) and also to $X(3)$. The method of proof is to use the algorithm given by Selman Akbulut and Rob Kirby in [AK] for drawing Kirby calculus diagrams of 4-manifolds given as branched covers, and to demonstrate that the branched covers match handlebody descriptions of the corresponding Lefschetz fibration.[6]

If we compare the g-fold fiber sum $\#_F(gX(1))$ with $X(2)$, we have two genus g fibrations with the same number $4g(2g + 1)$ of singular fibers. While the fibrations can easily be seen to be inequivalent–their monodromy representations have different images in M_g–the underlying smooth 4-manifolds could conceivably be diffeomorphic for $g \equiv 1$ or $2 \mod 4$.

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[5]In the case $Y = \mathbf{CP}^2 \# k\overline{\mathbf{CP}^2}$, some of these fibers are immersed spheres with a transverse self-intersection. In fact, these projections comprise exactly the collection of all (necessarily non-relatively minimal) genus 0 Lefschetz fibrations.

[6]The Lefschetz fibration on $X(4)$ is also obtained as a branched cover, of $T^2 \times S^2$. Indeed, this example was discovered by Matsumoto by beginning with a branched cover, and using a computer calculation to show that the monodromy of the resulting Lefschetz fibration was given by equation (4).

The question of whether they are was first posed by Matsumoto (for $g = 2$ [Ki]; the generalization to arbitrary g was posed by Hisaaki Endo [E]), and answered in the negative for $g = 2$ by the author in [F1]. (The argument extends easily to all g.) They can be distinguished smoothly by using the branched cover descriptions of the Proposition, from which it follows that $\#_F(gX(1))$ is irreducible, whereas $X(2) = Y \# \overline{\mathbf{CP}^2}$, for a 4-manifold Y.

Each of the previous examples has the special property that the vanishing cycles appear in a symmetric fashion on Σ_g, which is necessary for the fibration to restrict to the singular fibers as a branched cover. This motivates the following definition.

Definitions. The *hyperelliptic mapping class group* H_g is the subgroup of M_g of classes which commute with the class of the involution $\iota : \Sigma_g \to \Sigma_g$ given by the 180° rotation pictured in Figure 11. A genus g Lefschetz fibration is *hyperelliptic* if the monodromy about
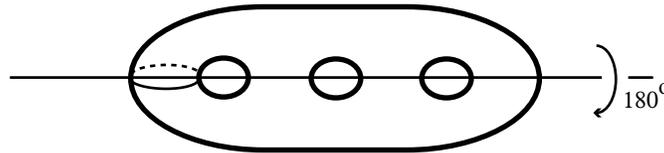

Figure 11: hyperelliptic involution ι

each of its singular points (with respect to a system of arcs s_i) is contained in H_g.

It is not hard to see that a Lefschetz fibration is hyperelliptic if and only if each of the vanishing cycles $\gamma_1, \ldots, \gamma_\mu$ can be isotoped so that $\iota(\gamma_i) = \gamma_i$. In particular, each of the examples $X(1), \ldots, X(4)$ is hyperelliptic. Being hyperelliptic means that each individual fiber is obtained as a 2-fold branched cover of S^2, and the following theorem, proven independently by the author and by Bernd Siebert and Gang Tian shows that this local symmetry extends to the global Lefschetz fibration.

Theorem. *([F2]; [ST]) If $f : X \to S^2$ is a relatively minimal hyperelliptic Lefschetz fibration all of whose vanishing cycles are nonseparating curves, the X is obtained as a 2-fold branched cover of an S^2-bundle over S^2. If f includes s separating curves among its vanishing cycles, then X is obtained (after blowing down all exceptional spheres found in fibers) as a 2-fold branched cover of $\mathbf{CP}^2 \# (2s + 1)\overline{\mathbf{CP}^2}$, branched over an embedded surface.*

In addition, as every genus 2 Lefschetz fibration is hyperelliptic, a corollary of the above theorem is that every genus 2 Lefschetz fibration is obtained as a branched cover.

The two proofs of this Theorem are very different. In [F2], the branched covers are constructed by hand, using the handlebody description of X given in Section 3 to show that models of the branched cover for each fiber can be patched together to achieve the global

fibration as a branched cover; in [ST], a detailed analysis of the branch sets together with patching arguments borrowed from complex algebraic geometry are used. The approach of Siebert and Tian has the advantage that it can be used to show that the branch sets producing the Lefschetz fibration are symplectically embedded. In [ST], they conjecture that in fact every hyperelliptic Lefschetz fibration with only nonseparating vanishing cycles is complex.[7] Thus the existence of separating curves as vanishing cycles can be viewed as an obstruction to being complex–note that the example of Ozbacgi and Stipsicz is hyperelliptic but contains separating vanishing cycles.

It is natural to wonder to what extent an arbitrary non-hyperelliptic Lefschetz fibration can be represented as a branched cover, since in general a vanishing cycle need not have the symmetric appearance of those in our hyperelliptic examples. This difficulty can be circumvented of one works with irregular simple 3-fold branched covers.[8] For any genus g, there is a simple 3-fold branched cover $\pi : \Sigma_g \to S^2$, and any simple closed curve on Σ_g can be isotoped so that a singular fiber with it as a vanishing cycle is also a 3-fold covering of S^2. This leads to the following Theorem, which can be proven analogously to the previous by using the handlebody description of X induced from the Lefschetz fibration. (A similar theorem for Lefschetz fibrations whose fibers are surfaces with boundary has been proven by Andrea Loi and Riccardo Piergallini [LP].)

Theorem. *([F3]) Any Lefschetz fibration $f : X \to D^2$ all of whose vanishing cycles are nonseparating curves can be obtained as a simple 3-fold branched cover of $S^2 \times D^2$ branched over an embedded surface.*

Another thread of research relating Lefschetz fibrations and branched coverings comes from the work of Denis Auroux.

Theorem. *([A]) If X is a symplectic 4-manifold, then there exists a simple branched cover $X \to \mathbf{CP}^2$ branched over an immersed surface which is smooth except for a finite number of cusp singularities.*

Auroux' theorem is obtained by extending techniques from complex projective geometry to the symplectic category in order to construct a map $X \to \mathbf{CP}^2$ for any symplectic X. A careful analysis of this map shows that it is a simple branched cover of the sort described. Auroux and Ludmil Katzarkov have defined invariants of symplectic 4-manifolds by describing the branch sets in $\mathbf{CP}^2$ associated to this map as braids [AuK].

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[7]Bernd Siebert recently informed the author that he and Tian have proven this conjecture.

[8]Simple branched covers of surfaces are important constructions in low-dimensional topology, largely because they are adaptable to settings without any intrinsic symmetry, such as the non-hyperelliptic Lefschetz fibrations considered here. See [BE] for a very readable survey.

7 Other Results

We conclude by briefly mentioning some other recent results about Lefschetz fibrations.

Signatures of Lefschetz fibrations. It is a straightforward calculation using the handlebody description of Section 3 to see that the Euler characteristic of a 4-manifold X admitting a genus g Lefschetz fibration is given by $e(X) = 2(2 - 2g) + \mu$, where μ is the number of singular fibers. It is less straightforward, however, to calculate the signature $\sigma(X)$ from the description of the Lefschetz fibration on X.

For hyperelliptic Lefschetz fibrations, "local signature" formulas have been established by Matsumoto [Ma] (for $g = 2$) and Endo [E] (for arbitrary g). The signature of X may be calculated by summing together contributions to the signature from each singular fiber, each of whose contribution is determined solely by its diffeomorphism type. (Their formulas also follow from the branched cover techniques of Section 6.)

Signatures of arbitrary Lefschetz fibrations have been studied by Ozbagci, who gives an algorithm for computing the signature of X from a list of its vanishing cycles [O].

Estimates on Numbers of Singular Fibers. Several researchers, using a variety of ideas, have been able to find estimates on the number of singular fibers in a Lefschetz fibration in terms of the fiber genus g. The best results along these lines are summarized below.

Theorems. *Let X be a genus g Lefschetz fibration over S^2. Let n and s denote the number of singular fibers given by vanishing cycles about nonseparating and separating curves, respectively, so that $\mu = n + s$. Then the following estimates hold.*

(1.) ([L]) $n \geq g$;

(2.) ([St2]) $\mu \geq \frac{1}{5}(8g - 4)$.

We note that $n > 0$ is a consequence of (1.); in particular, no Lefschetz fibration can have only separating vanishing cycles. This was first proven by Smith [ABKP].

The assumption of base S^2 is quite necessary, as the following theorem of Ozbagci and Mustafa Korkmaz shows.

Theorem. *([KoOz]) There exists a genus g Lefschetz fibration over Σ_h with one singular fiber if and only if $g \geq 3$ and $h \geq 2$.*

Realizing Fundamental Groups as Lefschetz Fibrations. In [ABKP], a constructive proof is given that for any finitely presented group G, there exists a Lefschetz fibration $f : X \to S^2$ with $\pi_1(X) = G$. The fact that all finitely presented groups can be realized as

the fundamental group of a symplectic 4-manifold had earlier been established by Gompf [G1]. As the fundamental groups of Kahler complex surfaces are known to be restricted (for example, the rank of their abelianization must be even), these results further display the distinction between symplectic and complex 4-manifolds.

Lefschetz Fibrations and Stein Surfaces. A *Stein surface* is a complex surface S that admits a proper biholomorphic embedding into $\mathbf{C}^N$, for some N. These manifolds admit Morse functions $f : S \to [0, \infty)$, obtained for instance as the distance from points in S to a fixed generic point in $\mathbf{C}^N$. (Stein surfaces can, in fact, be characterized by the existence of these maps.) While a Stein surface as such can never be compact, by considering $f^{-1}([0, t])$ for a regular value t one obtains *compact Stein surfaces*, which can be thought of as Stein surfaces with boundary. The topology of compact Stein surfaces has become an increasingly active field lately, in large part because of their relationship (via their boundaries $f^{-1}(\{t\})$) to the topology of contact 3-manifolds. Particularly noteworthy has been work of Gompf who, building on work of Eliashberg, developed a version of Kirby calculus for working with Stein surfaces [G2]. We refer the reader to Chapter 11 of [GS] for an introduction to Stein surfaces.

The following theorem of Loi and Piergallini relates compact Stein surfaces and Lefschetz fibrations.

Theorem. *([LP]) If X is a compact Stein surface, then X admits a Lefschetz fibration over D^2 with bounded fibers.*

A different proof of this theorem has been given by Akbulut and Ozbagci, who show moreover that a compact Stein surface X admits infinitely many nonequivalent such Lefschetz fibrations [AO].

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