A smooth function with certain convexity properties

Sungjin Kim, Wai Yan Pong

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Abstract

We construct a smooth function with certain convexity properties answering a question left open in a previous paper by Pong and Raianu. Moreover, we show that a second derivative must be convex near a point of convexity unless it is a limit point of its zeros. This strengthens another result in the previous paper.

Key Words: Point of convexity of a function, function convex at a point, punctual convexity
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There are several notions of pointwise convexity in the literature [1, 2, 3, 4, 6], and [7]. Pong and Raianu studied their relationships in [5]. Among the many examples given in that article is a family of C^n $(n \geq 1)$ functions that are p-convex but not convex at a point of convexity. They also showed that no analytic example is possible since a function must be convex near an analytic point of convexity [5, Theorem 1]. However, smooth examples indeed exist as we will construct one in this article.

We start by recalling various notions of pointwise convexity that will be discussed. Let $\Psi(x_0, x_1, x_2)$ be the second difference quotient of a function f, i.e.

$$\Psi(x_0, x_1, x_2) = \frac{\varphi(x_2, x_0) - \varphi(x_1, x_0)}{x_2 - x_1}$$

where $\varphi(x, y) = (f(x) - f(y))/(x - y)$. Near x_0, by that we mean on some open interval containing x_0, if

1. $\Psi(x_0, x_1, x_2) \geq 0$ for all x_1, x_2 on opposite sides of x_0 then x_0 **is a point of convexity of** f.

2. $\Psi(x_0, x_1, x_2) \geq 0$ for all x_1, x_2 on the same side of x_0 then f **is convex at** x_0.

3. whenever x_1, x_2 are on opposite sides of x_0 with $x_0 + x_0' = x_1 + x_2$,

$$\begin{cases} \Psi(x_1, x_0, x_0') + \Psi(x_2, x_0, x_0') \geq 0 & \text{if } x_0 \neq x_0'; \text{or} \\ \Psi(x_0, x_1, x_2) \geq 0 & \text{if } x_0 = x_0'. \end{cases}$$

   then f **is p-convex at** x_0.

Note that being convex at x_0 means $\varphi(x, x_0)$ is increasing on either side of x_0. When f is differentiable, that means $\varphi'(x, x_0) \geq 0$ on either side of x_0. Also, by Theorem 2 of [1], f is p-convex at x_0 if $f'(x_1) \leq f'(x_0) \leq f'(x_2)$ whenever $x_1 \leq x_0 \leq x_2$ are two points near x_0. In particular, to show that an even differentiable function f is p-convex at 0, it suffices to check that $f'(x) > 0$ for positive x because by the fact that f' is odd, $f'(x) < 0$ for negative x and by the intermediate value property of derivatives (Darboux's Theorem) $f'(0) = 0$. Incidentally, when these conditions are met, 0 is also a local minimum of f, and hence is a point of convexity of f. We will construct a C^∞ function f and use the above criteria above to verify that

1. 0 is a point of convexity of f,

2. f is p-convex at 0, and

3. f is not convex at 0.

Example 0.1. *There is an even C^∞ function $f : \mathbb{R} \to \mathbb{R}$ such that $xf'(x) > 0$ for $x > 0$, $f(0) = 0$, and $\frac{d}{dx}\left(\frac{f(x)}{x}\right)$ changes sign infinitely many times in any open interval containing* 0*.*

Proof. Consider the smooth bump functions:

$$b_+(x) = \begin{cases} e^{-\frac{1}{(x-5)(9-x)}} & \text{if } 5 < x < 9 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$b_-(x) = \begin{cases} e^{-\frac{1}{(x-9)(10-x)}} & \text{if } 9 < x < 10 \\ 0 & \text{otherwise.} \end{cases}$$

The maximum of b_+ is $e^{-\frac{1}{4}}$ and the maximum of b_- is e^{-4}. We have

$$\int_9^{10} b_-(x)\,\mathrm{d}x < e^{-4} < \frac{1}{2}(8-6)e^{-\frac{1}{4}} < (8-6)e^{-\frac{1}{3}} < \int_5^9 b_+(x)\,\mathrm{d}x,$$

and since $40e^{-4} < e^{-\frac{1}{4}}$, we have

$$40\int_9^{10} b_-(x)\,\mathrm{d}x < \int_5^9 b_+(x)\,\mathrm{d}x. \tag{1}$$

We combine the two bump functions above into one up-down bump function:

$$b(x) = b_+(x) - b_-(x),$$

and we use (1) to obtain

$$B := \int_5^{10} b(x)\,\mathrm{d}x > \frac{39e^{-\frac{1}{4}}}{40} > 0. \tag{2}$$

Now, for $x > 0$, consider the sum of infinitely many shrunken copies of the up-down bump function:

$$g(x) = \sum_{n=0}^{\infty} \frac{b(2^n x)}{2^{n^2}}.$$

For $x \geq 10$, $g(x) = 0$, and for $0 < x < 10$, if $m \geq 0$ is the unique integer such that $5/2^m < x \leq 10/2^m$ we have:

$$g(x) = \begin{cases} \frac{b(2^m x)}{2^{m^2}} > 0 & \text{if } 2^m x < 9 \\ \frac{b(2^m x)}{2^{m^2}} \geq \frac{-e^{-4}}{2^{m^2}} & \text{if } 2^m x \geq 9. \end{cases}$$

Then we extend g to $\mathbb{R}$ by defining $g(0) = 0$ and requiring $g(x) = g(-x)$ for all $x < 0$. Near any positive x, g is just the sum of at most two shrunken copies of b and so g is C^∞ at x and

$$g^{(k)}(x) = \sum_{n=0}^{\infty} \frac{b^{(k)}(2^n x)}{2^{n^2 - kn}} \quad (x > 0).$$

To show that g is C^∞ at 0, assume $g^{(k)}(0) = 0$ for some $k \geq 0$ (the case $k = 0$ is true by definition). For any $\varepsilon > 0$, let $n_0 = n_0(\varepsilon, k) > k + 1$ be sufficiently large so that

$$\frac{B_k}{2^{n_0^2 - n_0(k+1)}} < \varepsilon.$$

where B_k bounds $|b^{(k)}(x)|$ everywhere. Then for any $0 < x \leq 10/2^{n_0}$, $5/2^m < x \leq 10/2^m$ for some $m \geq n_0$. Thus,

$$\left|\frac{g^{(k)}(x) - g^{(k)}(0)}{x - 0}\right| = \left|\sum_{n=0}^{\infty} \frac{b^{(k)}(2^n x)}{2^{n^2-kn}x}\right| \leq \frac{B_k}{2^{m^2-mk}} \cdot \frac{2^m}{5} < \frac{B_k}{2^{n_0^2-n_0(k+1)}} < \varepsilon.$$

This shows that the right derivative of $g^{(k)}(x)$ at 0 exists and is 0. For $x < 0$, $g^{(k)}(-x) = (-1)^k g^{(k)}(x)$ so the same argument show that the left-derivative of $g^{(k)}(x)$ at 0 is also 0. Thus, $g^{(k+1)}(0) = 0$ and so g is C^∞ at 0 by induction.

Next, we define f by $f(x) = xG(x)$ where $G(x) = \int_0^x g(s)\mathrm{d}s$. So $f(0) = 0$ and note that near 0, $\varphi'(x, 0) = (f(x)/x)' = g(x)$ changes sign on either side of 0. Hence f is not convex at 0.

By the uniform convergence of the series defining g, we have

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n^2}} \int_0^x b(2^n s)\ \mathrm{d}s.$$

The change of variable $2^n s = u$ yields,

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n^2+n}} \int_0^{2^n x} b(u)\ \mathrm{d}u.$$

Let $0 < x \leq 10$, let $m \geq 0$ be the unique integer such that $5/2^m < x \leq 10/2^m$. Then

$$g(x) = \frac{b(2^m x)}{2^{m^2}} \geq \frac{-e^{-4}}{2^{m^2}}, \tag{3}$$

Since $g(x) > 0$ for $5/2^m < x < 9/2^m$, we have $G(x) > 0$ in this range.

For $9/2^m \leq x \leq 10/2^m$, we have

$$G(x) \geq \sum_{n=m}^{\infty} \frac{B}{2^{n^2+n}} \geq \frac{B}{2^{m^2+m}} > 0, \tag{4}$$

where $B = \int_5^{10} b(x)\ \mathrm{d}x$. Since $G(x) = G(10)$ for $x > 10$, we conclude that $G(x) > 0$ and hence $f(x) > 0$ for all $x > 0$.

To prove that $f'(x) > 0$ for positive x, consider the logarithmic derivative of f,

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{g(x)}{G(x)}.$$

We prove that $f'(x)/f(x)$ is positive for $x > 0$ by checking how far $g(x)/G(x)$ can go in the negative direction. Note that $f'(x)/f(x)$ is positive when $g(x)$ is nonnegative, so we only need to consider those x's such that $g(x) < 0$. In particular, $9/2^m < x < 10/2^m$ and so by Equation (3) and (4),

$$\frac{g(x)}{G(x)} \geq \frac{-e^{-4}}{2^{m^2}} \cdot \frac{2^{m^2+m}}{B} = -\frac{e^{-4}2^m}{B}.$$

Using $10/x > 2^m$ and (2), it follows that

$$\frac{g(x)}{G(x)} \geq -e^{-4}\frac{10}{x} \cdot \frac{40e^{1/4}}{39} = -\frac{400}{39e^{15/4}x} > -\frac{1}{4x}.$$

Therefore,

$$\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{g(x)}{G(x)} \geq \frac{1}{x} - \frac{1}{4x} = \frac{3}{4x} > 0.$$

Hence, $f(x) = xG(x)$ satisfies 1, 2, and 3. □

We conclude this article with a stronger version of Proposition 6 of [5]. We realize the continuity assumption of the second derivative in the original statement can be removed when we revisited that paper.

Proposition 0.1. *Let f be twice differentiable near a point of convexity x_0 then f is convex near x_0 unless x_0 is a limit point of the zeros of f''.*

Proof. Suppose x_0 is not a limit point of the zeros of f'', that is, f'' is zero-free on some punctured neighborhood $I \setminus \{x_0\}$ of x_0. If $f'' > 0$ on I, then f is convex on I. If $f'' < 0$ on I, then $-f$ is convex on I and so $\varphi(x_1, x_0) \geq \varphi(x_2, x_0)$ for all $x_1 < x_2$. But since x_0 is a point of convexity of f, $\varphi(x_1, x_0) \leq \varphi(x_2, x_0)$ for all $x_1 < x_0 < x_2$. Therefore, $\varphi(x, x_0)$ must be constant on I (and the constant is $f'(x_0)$) hence f is linear and therefore also convex on I.

As a derivative, f'' satisfies the intermediate value property. So, if f'' changes sign on I at all, its sign must change across x_0 and $f''(x_0) = 0$ as x_0 is the only possible zero of f'' in I. Suppose f'' changes from negative to positive across x_0. Since x_0 is a point of convexity of f, for $x_1 < x_0 < x_2$,

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1} \leq \frac{f(x_2) - f(x_0)}{x_2 - x_0}.$$

Letting $x_2 \to x_0$, we conclude that for all $x_1 < x_0$,

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1} \leq f'(x_0).$$

So, $f'(u) \leq f'(x_0)$ for some $x_1 < u < x_0$ according to the mean value theorem. But this contradicts $f'' < 0$ on (x_1, x_0). Now if f'' changes from positive to negative across x_0, then one should let $x_1 \to x_0$ and conclude in a similar way that $f'(x_0) \leq f'(u)$ for some $x_0 < u < x_2$, contradicting $f'' < 0$ on (x_0, x_2). $\square$

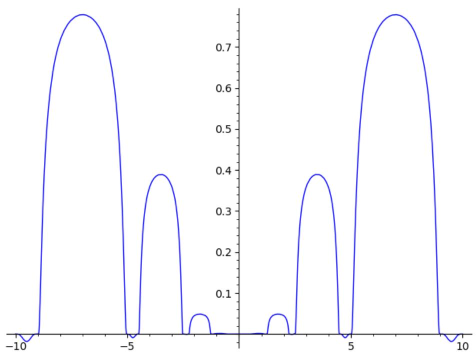

Figure 1: The function $g(x)$

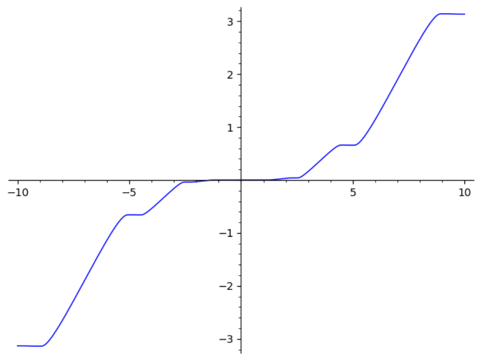

Figure 2: The function $G(x)$

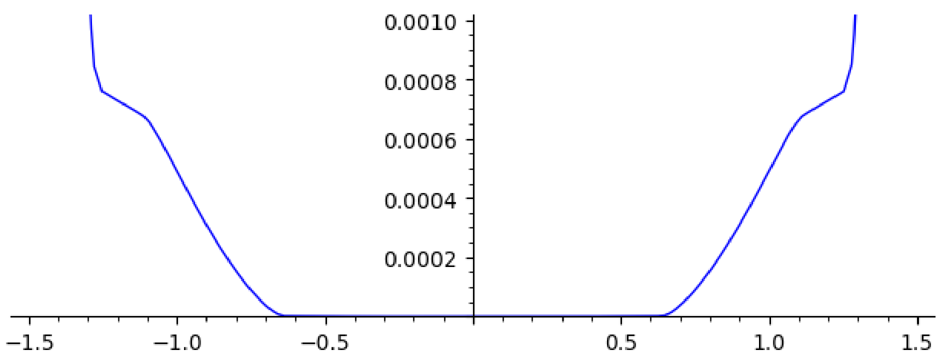

Figure 3: The function $f(x)$

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