THE HISTORY OF TITCHMARSH DIVISOR PROBLEM

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Let $\tau(n) = \sum_{d|n} 1$ be the divisor function, $a \neq 0$ be fixed integer. We define the following constants, where γ is the Euler-Mascheroni constant.

$$C_1(a) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{p|a} \left(1 - \frac{p}{p^2 - p + 1}\right)$$

$$C_2(a) = C_1(a) \left(\gamma - \sum_{p} \frac{\log p}{p^2 - p + 1} + \sum_{p|a} \frac{p^2 \log p}{(p-1)(p^2 - p + 1)}\right)$$

Theorem 1 (1931)**.** [T] *Under GRH for Dirichlet L-functions,*

$$\sum_{p \le x} \tau(p + a) = C_1(a)x + O\left(\frac{x \log\log x}{\log x}\right). \tag{1}$$

Theorem 2 (1963)**.** [L] *Unconditionally by dispersion method,*

$$\sum_{p \le x} \tau(p + a) = C_1(a)x + O\left(\frac{x \log\log x}{\log x}\right). \tag{2}$$

Halberstam(1967) [H] gave a simpler unconditional proof using Bombieri-Vinogradov theorem and Brun-Titchmarsh inequality.

Bombieri, Friedlander, and Iwaniec(1986) [BFI], independently by Fouvry(1984) [F] obtained more precise formula

Theorem 3. [BFI] *Let $A > 0$ be fixed.*

$$\sum_{n \le x} \Lambda(n)\tau(n + a) = C_1(a)x \log x + (2C_2(a) - C_1(a))x + O\left(\frac{x}{\log^A x}\right). \tag{3}$$

Using partial summation to above, we have

Corollary 1.

$$\sum_{p \le x} \tau(p + a) = C_1(a)x + 2C_2(a)\mathrm{Li}(x) + O\left(\frac{x}{\log^A x}\right). \tag{4}$$

This result heavily relies on Bombieri-Vinogradov type result without having absolute value in the sum.

Theorem 4. [BFI] *Let $A > 0$, then there is $B > 0$ depending on A such that*

$$\sum_{\substack{q \le x(\log x)^{-B} \\ (q,a)=1}} \left(\psi(x; q, a) - \frac{x}{\phi(q)}\right) \ll_{a,A} \frac{x}{\log^A x}. \tag{5}$$

By partial summation, we have

Corollary 2. *Let $A > 0$, then there is $B > 0$ depending on A such that*

$$\sum_{\substack{q \le x(\log x)^{-B} \\ (q,a)=1}} \left(\pi(x; q, a) - \frac{\pi(x)}{\phi(q)}\right) \ll_{a,A} \frac{x}{\log^A x}. \tag{6}$$

In view of this corollary, it looks like the moduli q came almost close to x. However, up to the full moduli $q \le x$, the estimate is very different. In fact, from the following lemma and Corollary 1:

Lemma 1.

$$\sum_{\substack{n\le x\\(n,a)=1}} \frac{1}{\phi(n)} = C_1(a)\log x + C_2(a) + O\left(\frac{\log x}{x}\right).$$

We obtain the following asymptotic for the full moduli.

Corollary 3.

$$\sum_{\substack{q\le x\\(q,a)=1}} \left(\pi(x;q,a) - \frac{\pi(x)}{\phi(q)}\right) = (C_2(a) - C_1(a))\mathrm{Li}(x) + O\left(\frac{x}{\log^2 x}\right). \tag{7}$$

For the primes in arithmetic progressions, A. T. Felix (2011) [Fe] proved that
Theorem 5. [Fe] *Fix integers $a \ne 0$ and $k \ge 1$. Then*

$$\sum_{\substack{p\le x\\p\equiv a \bmod k}} \tau\left(\frac{p-a}{k}\right) = \frac{c_k}{k}x + O\left(\frac{x}{\log x}\right), \tag{8}$$

where

$$c_k = C_1(a)\prod_{p|k}\left(1 + \frac{p-1}{p^2-p+1}\right).$$

Let $q' = \prod_{p|q} p = \mathrm{rad}(q)$. D. Fiorilli (2012) [Fi] obtained more precise formula. As a special case of [Fe, Theorem 2.4], we have
Theorem 6. [Fi] *Fix integers $a \ne 0$ and $q \ge 1$. Then*

$$\left|\sum_{|a|/q<m<x/q} \Lambda(qm+a)\tau(m) - M.T\right| \ll \frac{x}{\log^A x}, \tag{9}$$

where

$$M.T = \frac{x}{q}\left(C_1(a,q)\log x + 2C_2(a,q) + C_1(a,q)\log\left(\frac{(q')^2}{eq}\right)\right),$$
$$C_1(a,q) = \frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_{p|a}\left(1 - \frac{p}{p^2-p+1}\right)\prod_{p|q}\left(1 + \frac{p-1}{p^2-p+1}\right),$$

and

$$C_2(a,q) = C_1(a,q)\left(\gamma - \sum_p \frac{\log p}{p^2-p+1} + \sum_{p|a}\frac{p^2\log p}{(p-1)(p^2-p+1)} - \sum_{p|q}\frac{(p-1)p\log p}{p^2-p+1}\right).$$

We apply partial summation as before, then we have

Corollary 4.

$$\sum_{\substack{p\le x\\p\equiv a \bmod k}} \tau\left(\frac{p-a}{k}\right) = \frac{x}{k}C_1(a,k) + \frac{1}{k}\left(2C_2(a,k) + C_1(a,k)\log\left(\frac{(k')^2}{k}\right)\right)\mathrm{Li}(x) + O\left(\frac{x}{\log^A x}\right). \tag{10}$$

We write $C_1 = C_1(1)$ and $C_2 = C_2(1)$. In 2015, Sary Drappeau [D] obtained a power-saving error term under the GRH.

Theorem 7. [D] *Assume the GRH. For some $\delta > 0$, we have*

$$\sum_{n\le x} \Lambda(n)\tau(n-1) = C_1 x\log x + (2C_2 - C_1)x + O\left(x^{1-\delta}\right). \tag{11}$$

It would be natural to consider similar problems for k-divisor functions $\tau_k(n)$. Also in the paper [D], it was mentioned that current methods are not sufficient to obtain asymptotic formulas of $\sum_{p\le x}\tau_k(p-1)$ for $k\ge 3$. On the other hand, in an expository note by D. Koukoulopoulos (2015) [K, Exercise 4.3.2],

Theorem 8. *Unconditionally, we have*

$$\sum_{p\le x}\tau_3(p+a)\asymp x\log x\prod_{p|a}\left(1-\frac{1}{p}\right)^2. \tag{12}$$

Assuming Elliott-Halberstam Conjecture (EH), there is an absolute constant $C(a)$ such that

$$\sum_{p\le x}\tau_3(p+a)=C(a)x\log x+O(x). \tag{13}$$

References

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