THE AVERAGE NUMBER OF DIVISORS OF THE EULER FUNCTION

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Abstract. The upper bound and the lower bound of average numbers of divisors of Euler Phi function and Carmichael Lambda function are obtained by Luca and Pomerance (see [LP]). We improve the lower bound and provide a heuristic argument which suggests that the upper bound given by [LP] is indeed close to the truth.

1. Introduction

[1] Let $n \geq 1$ be an integer. Denote by $\phi(n)$, $\lambda(n)$, the Euler Phi function and the Carmichael Lambda function, which output the order and the exponent of the group $(\mathbb{Z}/n\mathbb{Z})^*$ respectively. We use p(or p_i), q(or q_i) to denote the prime divisors of n and $\phi(n)$ respectively. Then it is clear that $\lambda(n)|\phi(n)$ and the set of prime divisors q of $\phi(n)$ and that of $\lambda(n)$ are identical. Let $n = p_1^{e_1} \cdots p_r^{e_r}$ be a prime factorization of n. Then we can compute $\phi(n)$ and $\lambda(n)$ as follows:

$$\phi(n) = \prod_{i=1}^{r} \phi(p_i^{e_i}), \text{ and } \lambda(n) = \operatorname{lcm}\left(\lambda(p_1^{e_1}), \dots, \lambda(p_r^{e_r})\right)$$

where $\phi(p_i^{e_i}) = p_i^{e_i-1}(p_i - 1)$ and $\lambda(p_i^{e_i}) = \phi(p_i^{e_i})$ if $p_i > 2$ or $p_i = 2$ and $e_i = 1, 2$, and $\lambda(2^e) = 2^{e-2}$ if $e \geq 3$.

From the work of Hardy and Ramanujan [HR], it is well known that the normal order of $\tau(n)$ is $(\log n)^{\log 2 + o(1)}$. On the other hand, the average order $\frac{1}{x}\sum_{n\leq x} \tau(n)$ is known to be $\log x + O(1)$ which is somewhat larger than the normal order. For $\tau(\lambda(n))$ and $\tau(\phi(n))$, the normal orders of these follows from [EP] that they are $2^{\left(\frac{1}{2}+o(1)\right)(\log\log n)^2}$. On the contrary, the work of Luca and Pomerance [LP] showed that their average order is significantly larger than the normal order. Define $F(x) = \exp\left(\sqrt{\frac{\log x}{\log\log x}}\right)$. In [LP, Theorem 1,2], they proved that

$$F(x)^{b_1+o(1)} \leq \frac{1}{x}\sum_{n\leq x} \tau(\lambda(n)) \leq \frac{1}{x}\sum_{n\leq x} \tau(\phi(n)) \leq F(x)^{b_2+o(1)}$$

as $x \to \infty$, where $b_1 = \frac{1}{7}e^{-\gamma/2}$ and $b_2 = 2\sqrt{2}e^{-\gamma/2}$.

In this paper we are able to raise the constant b_1 so that it is almost b_2, differing only by a factor $\sqrt{2}$. Here, we take advantage of the inequalities of Bombieri-Vinogradov type regarding primes in arithmetic progression (see [BFI, Theorem 9], also [F, Theorem 2.1]). In this paper, we apply the following version which can be obtained from [F, Theorem 2.1]: For $(a, n) = 1$, we write $E(x; n, a) := \pi(x; n, a) - \frac{\pi(x)}{\phi(n)}$. Let $0 < \lambda < 1/10$. Let $R \leq x^\lambda$. For some $B = B(A) > 0$, $M = \log^B x$, and $Q = x/M$,

$$\sum_{\substack{r\leq R\\(r,a)=1}} \left| \sum_{\substack{q\leq \frac{Q}{r}\\(q,a)=1}} E(x; qr, a) \right| \ll_{A,\lambda} x \log^{-A} x.$$

In fact, [F, Theorem 2.1] builds on [BFI, Theorem 9] and obtains a more accurate estimate, but we only need the above form for our purpose. Note that one of the important differences between [BFI, Theorem 9] and [F, Theorem 2.1] is the presence of $\frac{Q}{r}$ in the inner sum. This will be essential in the proof of our lemmas (see Lemma 2.2 and 2.3).

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It is interesting to note that one of these improvements is related to a Poisson distribution that we can obtain from prime numbers. Another point of improvement comes from the idea in the proof of Gauss' Circle Problem.

Theorem 1.1. *As $x \to \infty$, we have*

$$\sum_{n\le x}\tau(\phi(n)) \ge \sum_{n\le x}\tau(\lambda(n)) \ge x\exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log\log x}}(1+o(1))\right).$$

It is clear from $\lambda(n)|\phi(n)$ that $\sum_{n\le x}\tau(\lambda(n)) \le \sum_{n\le x}\tau(\phi(n))$. A natural question to ask is how large is the latter compared to the former. Luca and Pomerance proved in [LP, Theorem 2] that

$$\frac{1}{x}\sum_{n\le x}\tau(\lambda(n)) = o\left(\max_{y\le x}\frac{1}{y}\sum_{n\le y}\tau(\phi(n))\right).$$

Moreover, they mentioned that a stronger statement

$$\frac{1}{x}\sum_{n\le x}\tau(\lambda(n)) = o\left(\frac{1}{x}\sum_{n\le x}\tau(\phi(n))\right)$$

is probably true, but they did not have the proof. Here, we prove that this statement is indeed true. As in the proof of [LP, Theorem 2], we take advantage of the fact that prime 2 appears rarely in the factorization of $\lambda(n)$ than in the factorization of $\phi(n)$.

Theorem 1.2. *As $x \to \infty$, we have*

$$\sum_{n\le x}\tau(\lambda(n)) = o\left(\sum_{n\le x}\tau(\phi(n))\right).$$

Finally, we give a heuristic argument suggests that the constant in the upper bound is indeed optimal. Here, we try to extend the method in the proof of Theorem 1.1 by devising a binomial distribution model. However, we were unable to prove it. The main difficulty is due to the short range of u ($u < \log^{A_1} x$) in the lemmas (see Lemma 2.1, 2.3, Corollary 2.1, and 2.2).

Conjecture 1.1. *As $x \to \infty$, we have*

$$\sum_{n\le x}\tau(\lambda(n)) = x\exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log\log x}}(1+o(1))\right).$$

Throughout this paper, x is a positive real number, n, k are positive integers, and p, q are prime numbers. We use Landau symbols O and o. Also, we write $f(x) \asymp g(x)$ for positive functions f and g, if $f(x) = O(g(x))$ and $g(x) = O(f(x))$. We will also use Vinogradov symbols $\ll$ and $\gg$. We write the iterated logarithms as $\log_2 x = \log\log x$ and $\log_3 x = \log\log\log x$. The notations (a, b) and $[a, b]$ mean the greatest common divisor and the least common multiple of a and b respectively. We write $P_z = \prod_{p\le z} p$. We also use the following restricted divisor functions:

$$\tau_z(n) := \prod_{\substack{p^e||n\\ p>z}}\tau(p^e),\quad \tau_{z,w}(n) := \prod_{\substack{p^e||n\\ z<p\le w}}\tau(p^e),\quad \text{and}\quad \tau_z'(n) := \prod_{\substack{p^e||n\\ p\le z}}\tau(p^e).$$

Moreover, for $n > 1$, denote by $p(n)$ the smallest prime factor of n.

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2. Lemmas

The following lemma is [LP, Lemma3] with a slightly relaxed z, and it is essential toward proving the theorem. This is stated and proved with the Chebyshev functions $\psi(x) := \sum_{n\le x} \Lambda(n)$ and $\psi(x;q,a) := \sum_{n\le x,\ n\equiv a \bmod q} \Lambda(n)$ in [LP2]. Here, we use the prime counting functions $\pi(x) := \sum_{p\le x} 1$ and $\pi(x;q,a) := \sum_{p\le x,\ p\equiv a \bmod q} 1$ instead. We are allowed to do these replacements by applying the partial summation.

Lemma 2.1. *Let $0<\lambda<\frac{1}{10}$. Assume that $z \le \lambda \log x$. Then for any $A>0$, there is $B=B(A)>0$ such that for $M=\log^B x$, and $Q=\frac{x}{M}$,*

$$E_z(x) := \sum_{r|P_z} \mu(r) \sum_{\substack{n\le Q\\ r|n}} \left(\pi(x;n,1) - \frac{\pi(x)}{\phi(n)}\right) \ll_{A,\lambda} \frac{x}{\log^A x}. \tag{1}$$

Let $0<\lambda<\frac{1}{10}$. Assume that u is a positive integer with $p(u)>z$, $u<(\log x)^{A_1}$ and $\tau(u)<A_1$. Then for any $A>0$, there is $B=B(A,A_1)>0$ such that for $M=\log^B x$, and $Q=\frac{x}{M}$,

$$E_{u,z}(x) := \sum_{r|P_z} \mu(r) \sum_{\substack{n\le Q\\ r|n}} \left(\pi(x;[u,n],1) - \frac{\pi(x)}{\phi([u,n])}\right) \ll_{A,A_1,\lambda} \frac{x}{\log^A x}. \tag{2}$$

Proof of (1). For $(a,n)=1$, we write $E(x;n,a) := \pi(x;n,a) - \frac{\pi(x)}{\phi(n)}$. If $r|P_z$, we have by the Prime Number Theorem, $r \le R := P_z = \exp(z+o(z)) \le x^{\lambda'}$ with $0<\lambda'<1/10$. By partial summation and diadically applying [F, Theorem 2.1], we have for $B=B(A)>0$, $M=\log^B x$, and $Q=x/M$,

$$\sum_{\substack{r\le R\\ (r,a)=1}} \left| \sum_{\substack{q\le \frac{Q}{r}\\ (q,a)=1}} E(x;qr,a) \right| \ll_{A,\lambda} \frac{x}{\log^A x}. \tag{3}$$

Taking $a=1$ and $|\mu(r)|\le 1$, (1) follows. □

Proof of (2). Let $d\le x^\epsilon$ so that $dR \le x^{\lambda'}$ with $0<\lambda'<1/10$. By (3), there exist $B=B(A)>0$ such that we have for $M=\log^B x$ and $Q=x/M$,

$$\sum_{r\le R} \left| \sum_{q\le \frac{Q}{r}} E(x;dqr,1) \right| = \sum_{\substack{r\le dR\\ r\equiv 0 \bmod d}} \left| \sum_{q\le \frac{Q}{r}} E(x;qr,1) \right| \le \sum_{r\le dR} \left| \sum_{q\le \frac{Q}{r}} E(x;qr,1) \right| \ll_{A,\lambda} \frac{x}{\log^A x}. \tag{4}$$

By $(u,r)=1$, we have $[u,n]=[u,qr]=r[u,q]=ruq/(u,q)$. We partition the set of $q\le\frac{Q}{r}$ as $\bigcup_{d|u} A_d$, where $q\in A_d$ if and only if $(u,q)=d$. Let $B_{Q,d} = \left\{q\le \frac{Q}{r} : q\equiv 0 \bmod d\right\}$. By inclusion-exclusion, we have for any $d|u$,

$$\sum_{q\in A_d} E\left(x;\frac{ruq}{d},1\right) = \sum_{s|\frac{u}{d}} \mu(s) \sum_{q\in B_{Q,ds}} E\left(x;\frac{ruq}{d},1\right).$$

It is clear that

$$\sum_{q\in B_{Q,ds}} E\left(x;\frac{ruq}{d},1\right) = \sum_{q\in B_{\frac{uQ}{d},us}} E(x;qr,1).$$

Since $r\le R := P_z < x^{\lambda'}$ with $\lambda' < \frac{1}{10}$, $\frac{uQ}{d} \le Q\log^{A_1} x$, and $us < \log^{2A_1} x < x^\epsilon$, we have by (4),

$$\sum_{r\le R} \left| \sum_{q\in B_{\frac{uQ}{d},us}} E(x;qr,1) \right| \ll_{A,A_1,\lambda} \frac{x}{\log^A x}$$

with a suitable choice of $B = B(A, A_1)$. Then

$$\sum_{r\le R}\left|\sum_{q\in A_d} E\left(x;\frac{ruq}{d},1\right)\right| = \sum_{r\le R}\left|\sum_{s|\frac{u}{d}}\mu(s)\sum_{q\in B_{Q,ds}} E\left(x;\frac{ruq}{d},1\right)\right|$$
$$\le \sum_{s|\frac{u}{d}}\sum_{r\le R}\left|\sum_{q\in B_{Q,ds}} E\left(x;\frac{ruq}{d},1\right)\right|$$
$$\ll_{A,A_1,\lambda} \tau\left(\frac{u}{d}\right)\frac{x}{\log^A x}.$$

Thus, summing over $d|u$, we have

$$\left|\sum_{r|P_z}\mu(r)\sum_{q\le\frac{Q}{r}} E(x;[u,qr],1)\right| \le \sum_{d|u}\sum_{r\le R}\left|\sum_{q\in A_d} E\left(x;\frac{ruq}{d},1\right)\right|$$
$$\ll_{A,A_1,\lambda} (\tau(u))^2\frac{x}{\log^A x} \ll_{A,A_1,\lambda} \frac{x}{\log^A x}.$$

Thus, we have the result (2). □

The following is [LP, Lemma 5] with a slightly relaxed z.

Lemma 2.2. *Let $0<\lambda<\frac{1}{10}$, and $1<z\le\lambda\log x$. Let $c_1 = e^{-\gamma}$. Then we have*

$$R_z(x) := \sum_{p\le x}\tau_z(p-1) = c_1\frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right), \tag{5}$$

and for $1<z\le\frac{\log x}{\log_2^2 x}$,

$$S_z(x) := \sum_{p\le x}\frac{\tau_z(p-1)}{p} = c_1\frac{\log x}{\log z} + O\left(\frac{\log x}{\log^2 z}\right). \tag{6}$$

Proof of (5). Take $A = 2$ and the corresponding $B(A)$ and M in Lemma 2.1(1). Then by inclusion-exclusion,

$$R_z(x) = \sum_{d\in D_z(x)}\pi(x;d,1) = \sum_{d\in D_z\left(\frac{x}{M}\right)}\pi(x;d,1) + \sum_{r|P_z}\mu(r)\sum_{\frac{x}{rM}<q\le\frac{x}{r}}\pi(x;qr,1) = R_1 + R_2,\ \text{say.}$$

By [LP, Lemma 4] and Lemma 2.1(1),

$$R_1 = \sum_{d\in D_z\left(\frac{x}{M}\right)}\frac{\pi(x)}{\phi(d)} + \sum_{r|P_z}\mu(r)\sum_{q\le\frac{x}{rM}} E(x;qr,1) = c_1\frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right) + O\left(\frac{x}{\log^2 x}\right).$$

By divisor-switching technique and Brun-Titchmarsh inequality as in [LP2], we have

$$R_2 \ll \sum_{r|P_z}\sum_{k\le M}\pi(x;rk,1) \ll \sum_{r|P_z}\sum_{k\le M}\frac{x}{\phi(rk)\log x} \ll \frac{x\log z\log M}{\log x} \ll \frac{x}{\log^2 z}.$$

Therefore, (5) follows. □

Proof of (6). By partial summation,

$$S_z(x) = \frac{R_z(t)}{t}\Big|_2^x + \int_2^x \frac{R_z(t)}{t^2}dt.$$

We split the integral at $z = \lambda\log t$. Then by (4),

$$\int_{z\le\lambda\log t}\frac{R_z(t)}{t^2}dt = \int_{e^{z/\lambda}}^x\left(c_1\frac{t}{\log z} + O\left(\frac{t}{\log^2 z}\right)\right)\frac{dt}{t^2} = c_1\frac{\log x}{\log z} + O\left(\frac{\log x}{\log^2 z}\right).$$

On the other hand, by the trivial bound $R_z(t) \ll t$,

$$\int_{z>\lambda \log t} \frac{R_z(t)}{t^2} dt \ll \int_2^{e^{z/\lambda}} t \frac{dt}{t^2} \ll z.$$

Since $z \log^2 z \ll \log x$, (6) follows. □

The following is [LP, Lemma 6] with a wider range of z. This relaxes the rather severe restriction $z \leq \frac{\sqrt{\log x}}{\log_2^6 x}$.

Lemma 2.3. *Let $1 \leq u \leq x$ be any positive integer. Then*

$$R_{u,z}(x) := \sum_{\substack{p \leq x \\ p \equiv 1 \bmod u}} \tau_z(p-1) \ll \frac{\tau(u)}{\phi(u)} x, \quad S_{u,z}(x) := \sum_{\substack{p \leq x \\ p \equiv 1 \bmod u}} \frac{\tau_z(p-1)}{p} \ll \frac{\tau(u)}{\phi(u)} \log x, \tag{7}$$

and $\phi(u)$ can be replaced by u if $p(u) > z$ and $\tau(u) < A_1$.

Assume that u is a positive integer with $p(u) > z$, $u < (\log x)^{A_1}$ and $\tau(u) < A_1$. Then for $z \leq \lambda \log x$,

$$R_{u,z}(x) = \frac{\tau(u)}{u} R_z(x) \left(1 + O\left(\frac{1}{\log z}\right)\right), \tag{8}$$

and for $z \leq \frac{\log x}{\log_2^2 x}$,

$$S_{u,z}(x) = \frac{\tau(u)}{u} S_z(x) \left(1 + O\left(\frac{1}{\log z}\right)\right). \tag{9}$$

Proof of (7). This is a uniform version of [Pe, Lemma 3.7]. We apply Dirichlet's hyperbola method as it was done in [Pe, Lemma 3.7]. First, we see that

$$R_{u,z}(x) \leq \sum_{\substack{p \leq x \\ p \equiv 1 \bmod u}} \tau(p-1) \leq \sum_{\substack{p \leq x \\ p \equiv 1 \bmod u}} \tau\left(\frac{p-1}{u}\right) \tau(u) \leq 2\tau(u) \sum_{k \leq \sqrt{\frac{x}{u}}} \pi(x; ku, 1).$$

Since the sum is zero for $x \leq u$, we may assume that $x > u$. By Brun-Titchmarsh inequality,

$$\pi(x; ku, 1) \leq \frac{2x}{\phi(ku) \log\left(\frac{x}{ku}\right)} \leq \frac{4x}{\phi(u)\phi(k) \log \frac{x}{u}}.$$

Thus, summing over k gives

$$\sum_{k \leq \sqrt{\frac{x}{u}}} \pi(x; ku, 1) \leq \frac{8x}{\phi(u)} \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d\phi(d)}.$$

Therefore, we have the result. The estimate for $S_{u,z}$ follows from partial summation.

We remark that for u with $p(u) > z$,

$$\frac{u\phi(d)}{\phi(ud)} = \prod_{p | u, p \nmid d} \left(1 - \frac{1}{p}\right)^{-1} = 1 + O\left(\frac{\tau(u)}{z}\right), \quad \frac{1}{\phi(u)} = \frac{1}{u} \prod_{p|u} \left(1 - \frac{1}{p}\right)^{-1} = \frac{1}{u}\left(1 + O\left(\frac{\tau(u)}{z}\right)\right).$$

Therefore, $\phi(u)$ can be replaced by u if $p(u) > z$ and $\tau(u) < A_1$. □

Proof of (8). We begin with

$$R_{u,z}(x) = \sum_{d \in D_z(x)} \pi(x; [u,d], 1).$$

Let $A > 0$ be a positive number that $\frac{x}{\log^A x} \ll \frac{\tau(u)}{u} \frac{x}{\log^2 x}$, and $B(A)$ and M be the corresponding parameters depending on A in Lemma 2.1(2). By inclusion-exclusion,

$$\sum_{d \in D_z(x)} \pi(x; [u,d], 1) = \sum_{d \in D_z\left(\frac{x}{M}\right)} \pi(x; [u,d], 1) + \sum_{r | P_z} \mu(r) \sum_{\frac{x}{rM} < q \leq \frac{x}{r}} \pi(x; [u, qr], 1) = R_1 + R_2, \quad \text{say}.$$

By Lemma 2.1(2), we have

$$R_1 = \sum_{d \in D_z\left(\frac{x}{M}\right)} \frac{\pi(x)}{\phi([u,d])} + \sum_{r|P_z} \mu(r) \sum_{q \le \frac{x}{rM}} E(x;[u,qr],1) = \sum_{d \in D_z\left(\frac{x}{M}\right)} \frac{\pi(x)}{\phi([u,d])} + O\left(\frac{\tau(u)}{u}\frac{x}{\log^2 x}\right).$$

The first sum is treated as follows:

$$\begin{aligned}
\sum_{d \in D_z\left(\frac{x}{M}\right)} \frac{\pi(x)}{\phi([u,d])} &= \sum_{d_1 \in D_z\left(\frac{x}{uM}\right)} \frac{\pi(x) N_{d_1}}{\phi(ud_1)} + O\left(\pi(x) \sum_{\substack{\frac{x}{uM} < d_1 \le \frac{x}{M} \\ p(d_1) > z}} \frac{\tau(u)}{\phi(ud_1)}\right) \\
&= \sum_{d_1 \in D_z\left(\frac{x}{uM}\right)} \frac{\pi(x) N_{d_1}}{\phi(ud_1)} + O\left(\pi(x) \frac{\tau(u)\log u}{\phi(u)\log z}\right) \\
&= \sum_{d_1 \in D_z\left(\frac{x}{uM}\right)} \frac{\pi(x) N_{d_1}}{\phi(ud_1)} + O\left(\frac{\tau(u)}{u}\frac{x}{\log^2 z}\right),
\end{aligned}$$

where $N_{d_1} = \left|\left\{d \in D_z\left(\frac{x}{M}\right) : [u,d] = ud_1\right\}\right|$. Since $N_{d_1} \le \tau(u)$ and $\phi(ud_1) \ge \phi(u)\phi(d_1)$, by [LP, Lemma 4],

$$\sum_{d_1 \in D_z\left(\frac{x}{uM}\right)} \frac{\pi(x) N_{d_1}}{\phi(ud_1)} \le \frac{\tau(u)}{\phi(u)}\left(c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right)\right).$$

Thus, we have the upper bound

$$\sum_{d_1 \in D_z\left(\frac{x}{uM}\right)} \frac{\pi(x) N_{d_1}}{\phi(ud_1)} \le \frac{\tau(u)}{u}\left(c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right)\right).$$

On the other hand, $N_{d_1} = \tau(u)$ if $(u,d_1) = 1$. Then, we may apply [LP, Lemma 4] since $P(u) \le \log^{A_1} x$, we obtain that

$$\begin{aligned}
\sum_{d_1 \in D_z\left(\frac{x}{uM}\right)} \frac{\pi(x) N_{d_1}}{\phi(ud_1)} &\ge \frac{\tau(u)}{u}\left(\sum_{\substack{d_1 \in D_z\left(\frac{x}{uM}\right) \\ (u,d_1)=1}} \frac{\pi(x)}{\phi(d_1)} + O\left(\frac{x}{\log^2 z}\right)\right) \\
&\ge \frac{\tau(u)}{u}\frac{\phi(u)}{u}\left(c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right)\right).
\end{aligned}$$

Thus, we have the lower bound

$$\sum_{d_1 \in D_z\left(\frac{x}{uM}\right)} \frac{\pi(x) N_{d_1}}{\phi(ud_1)} \ge \frac{\tau(u)}{u}\left(c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right)\right).$$

This shows that

$$R_1 = \frac{\tau(u)}{u}\left(c_1 \frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right)\right).$$

By divisor-switching technique and Brun-Titchmarsh inequality as in [LP2], we have

$$\begin{aligned} R_2 &\ll \sum_{r|P_z}\sum_{d|u}\sum_{s|\frac{u}{d}} \sum_{\substack{\frac{x}{rM}<q\le\frac{x}{r}\\ ds|q}} \pi\left(x;\frac{uqr}{d},1\right) \\ &\ll \sum_{r|P_z}\sum_{d|u}\sum_{s|\frac{u}{d}} \sum_{\frac{x}{dsrM}<q\le\frac{x}{dsr}} \pi\left(x;rusq,1\right) \\ &\ll \sum_{r|P_z}\sum_{d|u}\sum_{s|\frac{u}{d}} \sum_{k\le\frac{dM}{u}} \pi(x;rusk,1) \\ &\ll \sum_{r|P_z}\sum_{d|u}\sum_{s|\frac{u}{d}} \sum_{k\le\frac{dM}{u}} \frac{x}{\phi(rusk)\log x} \ll \tau(u)\frac{x\log z\log u\log M}{\phi(u)\log x} \ll \frac{\tau(u)}{u}\frac{x}{\log^2 z}. \end{aligned}$$

This completes the proof of (8). □

Proof of (9). We use (7) and (8), and apply partial summation as in (6). □

The following is used with inequality in [LP, Lemma 7]. Here, we obtain an equality that will be used frequently in this paper.

Lemma 2.4. *Let $0<\lambda<\frac{1}{10}$. Fix $a>1$ and an integer $0\le B<\infty$. We use $z=\lambda\log x$ for the formula for R_B and $z=\frac{\log x}{\log_2^2 x}$ for the formula for S_B. Let $I_a(x)=[z,z^a]$. Define*

$$\mathcal{U}_B=\{u: u \text{ is a positive square-free integer consisted of exactly } B \text{ prime divisors in } I_a(x)\}.$$

Then we have

$$R_B:=\sum_{u\in\mathcal{U}_B} R_{u,z}(x)=\frac{(2\log a)^B}{B!}R_z(x)\left(1+O\left(\frac{1}{\log z}\right)\right),$$

and

$$S_B:=\sum_{u\in\mathcal{U}_B} S_{u,z}(x)=\frac{(2\log a)^B}{B!}S_z(x)\left(1+O\left(\frac{1}{\log z}\right)\right).$$

Proof. We apply Lemma 2.3 with $u\in\mathcal{U}_B$. Note that $u\in\mathcal{U}_B$ satisfies the conditions for u in Lemma 2.3(8), (9). Then,

$$\begin{aligned} \sum_{u\in\mathcal{U}_B} R_{u,z}(x) &= \sum_{u\in\mathcal{U}_B}\frac{\tau(u)}{u}R_z(x)\left(1+O\left(\frac{1}{\log z}\right)\right) \\ &= \left(\frac{1}{B!}\left(\sum_{p\in I_a(x)}\frac{2}{p}\right)^B + O\left(\frac{1}{(B-2)!}\left(\sum_{p\in I_a(x)}\frac{4}{p^2}\right)\left(\sum_{p\in I_a(x)}\frac{2}{p}\right)^{B-2}\right)\right)R_z(x)\left(1+O\left(\frac{1}{\log z}\right)\right) \\ &= \left(\frac{1}{B!}\left(\sum_{p\in I_a(x)}\frac{2}{p}\right)^B + O\left(\frac{1}{z}\right)\right)R_z(x)\left(1+O\left(\frac{1}{\log z}\right)\right) \\ &= \frac{2^B}{B!}\left(\log\log z^a-\log\log z+O\left(\frac{1}{\log z}\right)\right)^B R_z(x)\left(1+O\left(\frac{1}{\log z}\right)\right) \\ &= \frac{(2\log a)^B}{B!}R_z(x)\left(1+O\left(\frac{1}{\log z}\right)\right). \end{aligned}$$

The result for S_B can be obtained similarly. □

Although we relaxed $z\le\frac{\sqrt{\log x}}{\log_2^6 x}$ to $z\le\frac{\log x}{\log_2^2 x}$, the range is still not enough for further use. We will see how this range can be relaxed to $\log^{\frac{1}{A}} x<z\le\log^A x$ in Lemma 2.5. A probability mass function of a Poisson distribution comes up as certain densities.

Lemma 2.5. *Let* $0 < \lambda < \frac{1}{10}$. *Fix* $a > 1$ *and an integer* $0 \le B < \infty$. *We use* $z = \lambda \log x$ *for the formula for* R'_B *and* $z = \frac{\log x}{\log_2^2 x}$ *for the formula for* S'_B. *Let* $I_a(x) = (z, z^a]$. *Define*

$$\tau_{z,z^a}(n) = \prod_{\substack{p^e || n \\ p \in I_a(x)}} \tau(p^e), \quad w_{z,z^a}(n) = |\{p|n \ : \ p \in I_a(x)\}|,$$

and

$$R'_B := \sum_{\substack{p \le x \\ w_{z,z^a}(p-1)=B}} \tau_z(p-1), \quad S'_B := \sum_{\substack{p \le x \\ w_{z,z^a}(p-1)=B}} \frac{\tau_z(p-1)}{p}.$$

Then as $x \to \infty$, *we have*

$$R'_B = \frac{(2\log a)^B}{B! a^2} R_z(x)(1+o(1)), \quad S'_B = \frac{(2\log a)^B}{B! a^2} S_z(x)(1+o(1)), \tag{10}$$

and we have

$$R_{z^a}(x) = \frac{1}{a} R_z(x)(1+o(1)), \quad S_{z^a}(x) = \frac{1}{a} S_z(x)(1+o(1)). \tag{11}$$

Proof of (10). We remark that by (7), (8), (9), the contribution of primes p such that $p-1$ is divisible by a square of a prime $q > z$ is negligible. In fact, those contributions to $R_z(x)$ and $S_z(x)$ are $O(R_z(x)/z)$ and $O(S_z(x)/z)$ respectively. Thus, we assume that $p-1$ is not divisible by square of any prime $q > z$. By Lemma 2.4 and inclusion-exclusion principle,

$$R'_B = R_B - \binom{B+1}{1} R_{B+1} + \binom{B+2}{2} R_{B+2} - \binom{B+3}{3} R_{B+3} + - \cdots .$$

Moreover, for any $k \ge 1$,

$$\sum_{j=0}^{2k-1} (-1)^j \binom{B+j}{j} R_{B+j} \le R'_B \le \sum_{j=0}^{2k} (-1)^j \binom{B+j}{j} R_{B+j}.$$

Then dividing by $R_z(x)$ gives

$$\sum_{j=0}^{2k-1} (-1)^j \binom{B+j}{j} \frac{R_{B+j}}{R_z(x)} \le \frac{R'_B}{R_z(x)} \le \sum_{j=0}^{2k} (-1)^j \binom{B+j}{j} \frac{R_{B+j}}{R_z(x)}.$$

By Lemma 2.4, we have

$$\frac{(2\log a)^B}{B!} \sum_{j=0}^{2k-1} (-1)^j \frac{(2\log a)^j}{j!} \left(1 + O\left(\frac{1}{\log z}\right)\right) \le \frac{R'_B}{R_z(x)} \le \frac{(2\log a)^B}{B!} \sum_{j=0}^{2k} (-1)^j \frac{(2\log a)^j}{j!} \left(1 + O\left(\frac{1}{\log z}\right)\right).$$

Taking $x \to \infty$, we have

$$\frac{(2\log a)^B}{B!} \sum_{j=0}^{2k-1} (-1)^j \frac{(2\log a)^j}{j!} \le \liminf_{x\to\infty} \frac{R'_B}{R_z(x)} \le \limsup_{x\to\infty} \frac{R'_B}{R_z(x)} \le \frac{(2\log a)^B}{B!} \sum_{j=0}^{2k} (-1)^j \frac{(2\log a)^j}{j!}.$$

Letting $k \to \infty$, we obtain

$$\lim_{x\to\infty} \frac{R'_B}{R_z(x)} = \frac{(2\log a)^B}{B! a^2}.$$

The result for S'_B can be obtained similarly. □

Proof of (11). As in the proof of (10), we assume that $p-1$ is not divisible by square of any prime $q > z$. Note that $\tau_z(p-1) = \tau_{z^a}(p-1)\tau_{z,z^a}(p-1)$. Let $0 \le B < \infty$ be a fixed integer. If $w_{z,z^a}(p-1) = B$ then $\tau_{z,z^a}(p-1) = 2^B$. Then we have by (10),

$$\sum_{\substack{p \le x \\ w_{z,z^a}(p-1)=B}} \tau_{z^a}(p-1) = \sum_{\substack{p \le x \\ w_{z,z^a}(p-1)=B}} \frac{\tau_z(p-1)}{2^B} = \frac{R'_B}{2^B} = \frac{(\log a)^B}{B! a^2} R_z(x)(1+o(1)).$$

Then by Lemma 2.4,

$$\begin{aligned}
\frac{R_{z^a}(x)}{R_z(x)} &= \sum_{j<B} \frac{(\log a)^j}{j!a^2}(1+o(1)) + \frac{1}{R_z(x)} \sum_{j\ge B} \frac{1}{2^j} \sum_{\substack{p\le x \\ w_{z,z^a}(p-1)=j}} \tau_z(p-1) \\
&= \sum_{j<B} \frac{(\log a)^j}{j!a^2}(1+o(1)) + O\left(\frac{1}{2^B R_z(x)} \sum_{\substack{p\le x \\ w_{z,z^a}(p-1)\ge B}} \tau_z(p-1) \right) \\
&= \sum_{j<B} \frac{(\log a)^j}{j!a^2}(1+o(1)) + O\left(\frac{R_B}{2^B R_z(x)} \right) \\
&= \sum_{j<B} \frac{(\log a)^j}{j!a^2}(1+o(1)) + O\left(\frac{(2\log a)^B}{2^B B!} \left(1+O\left(\frac{1}{\log z}\right)\right)\right).
\end{aligned}$$

Thus, both $\liminf\limits_{x\to\infty} \frac{R_{z^a}(x)}{R_z(x)}$ and $\limsup\limits_{x\to\infty} \frac{R_{z^a}(x)}{R_z(x)}$ are

$$\sum_{j\le B} \frac{(\log a)^j}{j!a^2} + O\left(\frac{(\log a)^B}{B!}\right)$$

and the constant implied in O does not depend on B. Therefore, letting $B\to\infty$, we obtain

$$\lim_{x\to\infty} \frac{R_{z^a}(x)}{R_z(x)} = \frac{1}{a}.$$

The result for $S_{z^a}(x)$ can be obtained similarly. □

Lemma 2.5 allows us to have an extended range of z, and the same method applied to $R_{u,z}(x)$, we can also extend range of z for $R_{u,z}(x)$ and $S_{u,z}(x)$.

Corollary 2.1. *Fix any $A>1$. Let $\log^{\frac{1}{A}} x < z \le \log^A x$. Then as $x\to\infty$, we have*

$$R_z(x) = c_1 \frac{x}{\log z}(1+o(1)), \quad S_z(x) = c_1 \frac{\log x}{\log z}(1+o(1)). \tag{12}$$

Assume that u is a positive integer with $p(u)>z$, $u<(\log x)^{A_1}$ and $\tau(u)<A_1$. Then as $x\to\infty$, we have

$$R_{u,z}(x) = \frac{\tau(u)}{u} R_z(x)(1+o(1)), \quad S_{u,z}(x) = \frac{\tau(u)}{u} S_z(x)(1+o(1)). \tag{13}$$

We apply Corollary 2.1 to obtain the following uniform distribution result:

Corollary 2.2. *Let $2\le v\le x$ and $r := (v^{\frac{3}{2}}\log v)^{-1}$. Suppose also that $r \ge \log^{-\frac{4}{5}} x$, $0\le\alpha\le\beta\le 1$, and $\beta-\alpha\ge r$. Then for $z\le \frac{\log x^r}{\log_2^2 x^r}$,*

$$\sum_{\alpha\le \frac{\log p}{\log x}<\beta} \frac{\tau_z(p-1)}{p} = (\beta-\alpha) S_z(x)\left(1+O\left(\frac{1}{\log z}\right)\right). \tag{14}$$

For $\log^{\frac{1}{A}} x < z \le \log^A x$, we have as $x\to\infty$,

$$\sum_{\alpha\le \frac{\log p}{\log x}<\beta} \frac{\tau_z(p-1)}{p} = (\beta-\alpha) S_z(x)\left(1+o(1)\right). \tag{15}$$

Assume that u is a positive integer with $p(u) > z$, $u < (\log x)^{A_1}$ and $\tau(u) < A_1$. Then we have for $z \le \frac{\log x^r}{\log_2^2 x^r}$,

$$\sum_{\substack{\alpha \le \frac{\log p}{\log x} < \beta \\ p \equiv 1 \bmod u}} \frac{\tau_z(p-1)}{p} = (\beta - \alpha)\frac{\tau(u)}{u} S_z(x)\left(1 + O\left(\frac{1}{\log z}\right)\right). \tag{16}$$

and for $\log^{\frac{1}{A}} x < z \le \log^A x$, we have as $x \to \infty$,

$$\sum_{\substack{\alpha \le \frac{\log p}{\log x} < \beta \\ p \equiv 1 \bmod u}} \frac{\tau_z(p-1)}{p} = (\beta - \alpha)\frac{\tau(u)}{u} S_z(x)\left(1 + o(1)\right). \tag{17}$$

Proof. By Lemma 2.2(5) and partial summation, we have for $\beta - \alpha \ge r$,

$$\begin{aligned} \sum_{\alpha \le \frac{\log p}{\log x} < \beta} \frac{\tau_z(p-1)}{p} &= \frac{R_z(t)}{t}\Big|_{x^\alpha}^{x^\beta} + \int_{x^\alpha}^{x^\beta} \frac{R_z(t)}{t^2} dt \\ &= c_1(\beta - \alpha)\frac{\log x}{\log z}\left(1 + O\left(\frac{1}{\log z}\right)\right) + O\left(\frac{1}{\log^2 z}\right). \end{aligned}$$

Clearly, $r \log x \gg 1$. Thus, the second O-term can be included in the first O-term. Then (14) follows.

Since $r \log x \ge \log^{\frac{1}{5}} x$, the range $\log^{\frac{1}{A}} x < z \le \log^A x$ can be obtained from taking powers of $\frac{\log x^r}{\log_2^2 x^r}$. We have by (12), as $x \to \infty$,

$$\begin{aligned} \sum_{\alpha \le \frac{\log p}{\log x} < \beta} \frac{\tau_z(p-1)}{p} &= \frac{R_z(t)}{t}\Big|_{x^\alpha}^{x^\beta} + \int_{x^\alpha}^{x^\beta} \frac{R_z(t)}{t^2} dt \\ &= c_1(\beta - \alpha)\frac{\log x}{\log z}\left(1 + o(1)\right) + o\left(\frac{1}{\log z}\right). \end{aligned}$$

Also, by $r \log x \gg 1$, the second o-term can be included in the first o-term. Therefore, (15) follows. Similarly, (16) follows from Lemma 2.3(8) and (17) follows from (13). □

We use $p_1, p_2, \ldots, p_v$ to denote prime numbers. We define the following multiple sums for $2 \le v \le x$:

$$\mathfrak{T}_{v,z}(x) := \sum_{p_1 p_2 \cdots p_v \le x} \frac{\tau_z(p_1 - 1)\tau_z(p_2 - 1)\cdots\tau_z(p_v - 1)}{p_1 p_2 \cdots p_v},$$

and for $\mathbf{u} = (u_1, \ldots, u_v)$ with $1 \le u_i \le x$,

$$\mathfrak{T}_{\mathbf{u},v,z}(x) := \sum_{\substack{p_1 p_2 \cdots p_v \le x \\ \forall_i,\ p_i \equiv 1 \bmod u_i}} \frac{\tau_z(p_1 - 1)\tau_z(p_2 - 1)\cdots\tau_z(p_v - 1)}{p_1 p_2 \cdots p_v},$$

Define $\mathbb{T}_v := \{(t_1, \ldots, t_v) : \forall_i,\ t_i \in [0,1],\ t_1 + \cdots + t_v \le 1\}$. We adopt the idea from Gauss' Circle Problem. Recall that $r = (v^{\frac{3}{2}} \log v)^{-1}$. Consider a covering of $\mathbb{T}_v$ by v-cubes of side-length r of the form:

Let $s_1, \ldots, s_v$ be nonnegative integers, let

$$B_{s_1,\ldots,s_v} := \{(t_1, \ldots, t_v) : \forall_i,\ r s_i \le t_i < r(s_i + 1)\}.$$

Let M_v be the set of those v-cubes lying completely inside $\mathbb{T}_v$. Then the sum $\mathfrak{T}_{v,z}(x)$ is over the primes satisfying:

$$\left(\frac{\log p_1}{\log x}, \ldots, \frac{\log p_v}{\log x}\right) \in \mathbb{T}_v.$$

Instead of the whole $\mathbb{T}_v$, we consider the contribution of the sum over primes satisfying:

$$\left(\frac{\log p_1}{\log x}, \ldots, \frac{\log p_v}{\log x}\right) \in \cup M_v,$$

which come from the v-cubes lying completely inside $\mathbb{T}_v$. We define

$$\mathfrak{S}_{v,z}(x) := \sum_{\left(\frac{\log p_1}{\log x},\dots,\frac{\log p_v}{\log x}\right)\in\cup M_v} \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v},$$

and similarly for $\mathbf{u}=(u_1,\cdots,u_v)$ with $1\le u_i\le x$,

$$\mathfrak{S}_{\mathbf{u},v,z}(x) := \sum_{\substack{\left(\frac{\log p_1}{\log x},\dots,\frac{\log p_v}{\log x}\right)\in\cup M_v \\ \forall_i,\ p_i\equiv 1 \bmod u_i}} \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v},$$

Let $v=\left\lfloor c\sqrt{\frac{\log x}{\log_2 x}}\right\rfloor$ for some positive constant c to be determined. Then v satisfies the conditions in Corollary 2.2. Then we have:

Lemma 2.6. *Let $\log^{\frac{1}{A}} x<z\le\log^A x$, then as $x\to\infty$,*

$$\mathfrak{S}_{v,z}(x)=\frac{1}{v!}S_z(x)^v(1+o(1))^v. \tag{18}$$

For $\mathbf{u}=(u_1,u_2,1,\dots,1)$ with $1\le u_i\le x$,

$$\mathfrak{S}_{\mathbf{u},v,z}(x)\ll\frac{\tau(u_1)\tau(u_2)}{\phi(u_1)\phi(u_2)}\mathfrak{S}_{v,z}(x)\log^k z, \tag{19}$$

where $0\le k\le 2$ is the number of u_i's that are not 1.

Assume that each u_i, $i=1,2$ is a positive integer with $p(u_i)>z$, $u_i<(\log x)^{A_1}$ and $\tau(u_i)<A_1$. Then as $x\to\infty$, we have

$$\mathfrak{S}_{\mathbf{u},v,z}(x)=\frac{\tau(u_1)\tau(u_2)}{u_1u_2}\mathfrak{S}_{v,z}(x)\left(1+o(1)\right). \tag{20}$$

Proof of (18). It is clear that

$$\mathrm{vol}\left((1-r\sqrt{v})\mathbb{T}_v\right)\le|M_v|\mathrm{vol}(B_{0,\dots,0})\le\mathrm{vol}(\mathbb{T}_v).$$

We have $\mathrm{vol}(\mathbb{T}_v)=\frac{1}{v!}$, $\mathrm{vol}(B_{0,\dots,0})=r^v$, and $\mathrm{vol}\left((1-r\sqrt{v})\mathbb{T}_v\right)=\frac{1}{v!}\left(1-r\sqrt{v}\right)^v$. Also, recall that $r:=\left(v^{\frac{3}{2}}\log v\right)^{-1}$. Then,

$$\frac{\frac{1}{v!}\left(1-\frac{1}{v\log v}\right)^v}{(v^{\frac{3}{2}}\log v)^{-v}}\le|M_v|\le\frac{\frac{1}{v!}}{(v^{\frac{3}{2}}\log v)^{-v}}.$$

On the other hand, by Corollary 2.2(15), the contribution of each v-cube $[\alpha_1,\beta_1]\times\cdots\times[\alpha_v,\beta_v]\subseteq[0,1]^v$ of side-length r to the sum is

$$\sum_{\forall_i,\ \alpha_i\le\frac{\log p_i}{\log x}<\beta_i}\frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v}=\left(\prod_{i=1}^v(\beta_i-\alpha_i)\right)S_z(x)^v(1+o(1))^v=r^vS_z(x)^v(1+o(1))^v.$$

Combining this with the bounds for $|M_v|$, we obtain the result. ∎

Proof of (19), (20). Let v and r be as defined in Corollary 2.2. We write (15) and (17) in the form of

$$\sum_{\alpha\le\frac{\log p}{\log x}<\beta}\frac{\tau_z(p-1)}{p}=(\beta-\alpha)S_z(x)\left(1+f_{\alpha,\beta}(x)\right), \tag{21}$$

and

$$\sum_{\substack{\alpha\le\frac{\log p}{\log x}<\beta\\ p\equiv 1 \bmod u}}\frac{\tau_z(p-1)}{p}=(\beta-\alpha)\frac{\tau(u)}{u}S_z(x)\left(1+g_{\alpha,\beta}(x)\right). \tag{22}$$

We note that there is a function $f(x)=o(1)$ such that uniformly for $0\le\alpha\le\beta\le 1$ and $\beta-\alpha\ge r$,

$$\max(|f_{\alpha,\beta}(x)|,|g_{\alpha,\beta}(x)|)\le f(x).$$

Then we can write

$$\sum_{\substack{\alpha\le\frac{\log p}{\log x}<\beta\\ p\equiv 1 \bmod u}}\frac{\tau_z(p-1)}{p}=(\beta-\alpha)\frac{\tau(u)}{u}S_z(x)\left(1+g_{\alpha,\beta}(x)\right)$$
$$=\frac{\tau(u)}{u}\sum_{\alpha\le\frac{\log p}{\log x}<\beta}\frac{\tau_z(p-1)}{p}\left(\frac{1+g_{\alpha,\beta}(x)}{1+f_{\alpha,\beta}(x)}\right)$$
$$=\frac{\tau(u)}{u}\sum_{\alpha\le\frac{\log p}{\log x}<\beta}\frac{\tau_z(p-1)}{p}\left(1+O(f(x))\right).$$

Consider any v-cube $[\alpha_1,\beta_1]\times\cdots\times[\alpha_v,\beta_v]\subseteq[0,1]^v$ of side-length r. Then by the above observation,

$$\sum_{\substack{\forall_i,\ \alpha_i\le\frac{\log p_i}{\log x}<\beta_i\\ p_i\equiv 1 \bmod u_i \text{ for } i=1,\ 2}}\frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v}$$
$$=\frac{\tau(u_1)\tau(u_2)}{u_1u_2}\sum_{\forall_i,\ \alpha_i\le\frac{\log p_i}{\log x}<\beta_i}\frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v}(1+O(f(x)))^2.$$

This proves (20). For the proof of (19), we use instead

$$\sum_{\substack{\alpha\le\frac{\log p}{\log x}<\beta\\ p\equiv 1 \bmod u}}\frac{\tau_z(p-1)}{p}=\frac{R_{u,z}(t)}{t}\Big|_{x^\alpha}^{x^\beta}+\int_{x^\alpha}^{x^\beta}\frac{R_{u,z}(t)}{t^2}dt$$
$$\ll\frac{\tau(u)}{\phi(u)}\left((\beta-\alpha)\log x+O(1)\right)\ll\frac{\tau(u)}{\phi(u)}(\beta-\alpha)\log x$$
$$\ll\frac{\tau(u)}{\phi(u)}(\beta-\alpha)S_z(x)\log z\ll\frac{\tau(u)}{\phi(u)}\sum_{\alpha\le\frac{\log p}{\log x}<\beta}\frac{\tau_z(p-1)}{p}\log z,$$

which follows from Lemma 2.3(7). □

We impose some restrictions on the primes $p_1,\ \ldots\ ,\ p_v$:

R1. $p_1,\ldots,p_v$ are distinct.
R2. For each i, $q^2\nmid p_i-1$ for any prime $q>z$.
R3. $q^2\nmid\phi(p_1\cdots p_v)$ for any prime $q>z^2$.

Recall that we chose

$$v=\left\lfloor c\sqrt{\frac{\log x}{\log_2 x}}\right\rfloor$$

for some positive constant c to be determined. Let ${\mathfrak{S}_{v,z}}^{(1)}(x)$ be the contribution of primes to $\mathfrak{S}_{v,z}(x)$ not satisfying R1. Note that if R1 is not satisfied, then some primes among $p_1,\ \ldots\ ,\ p_v$ are repeated. Then by

Lemma 2.6(18),

$$\begin{aligned}
\mathfrak{S}_{v,z}{}^{(1)}(x) &\ll \binom{v}{2}\left(\sum_{z<p\le x}\frac{\tau_z(p-1)^2}{p^2}\right)\mathfrak{S}_{v-2,z}(x)\\
&\ll v^2\frac{\log^3 z}{z}\frac{v(v-1)}{S_z(x)^2}\mathfrak{S}_{v,z}(x)\\
&\ll \frac{v^4\log^5 z}{z\log^2 x}\mathfrak{S}_{v,z}(x) \ll \frac{\log^3 z}{z}\mathfrak{S}_{v,z}(x).
\end{aligned}$$

Let $\mathfrak{S}_{v,z}{}^{(2)}(x)$ be the contribution of primes to $\mathfrak{S}_{v,z}(x)$ not satisfying R2. Note that if R2 is not satisfied, then $q^2|p_i-1$ for some primes p_i and $q>z$. Let $\mathbf{u}_{q^2} := (q^2,1,\dots,1)$. Suppose that $q^2|p_i-1$ for some p_i and $q>z^2$. Then the contribution of those primes to $\mathfrak{S}_{v,z}{}^{(2)}(x)$ is by (19),

$$\ll \sum_{q>z^2}\binom{v}{1}\mathfrak{S}_{\mathbf{u}_{q^2},v,z}(x) \ll \sum_{q>z^2}\frac{v}{\phi(q^2)}\mathfrak{S}_{v,z}(x)\log z \ll \sum_{q>z^2}\frac{v}{q^2}\mathfrak{S}_{v,z}(x)\log z \ll \frac{v}{z^2}\mathfrak{S}_{v,z}(x).$$

Suppose that $q^2|p_i-1$ for some p_i and $z<q\le z^2$, then we have by (20),

$$\ll \sum_{z<q\le z^2}\binom{v}{1}\mathfrak{S}_{\mathbf{u}_{q^2},v,z}(x) \ll \sum_{z<q\le z^2}\frac{v}{q^2}\mathfrak{S}_{v,z}(x) \ll \frac{v}{z\log z}\mathfrak{S}_{v,z}(x).$$

Thus, we have

$$\mathfrak{S}_{v,z}{}^{(2)}(x) \ll \frac{v}{z\log z}\mathfrak{S}_{v,z}(x).$$

Let $\mathfrak{S}_{v,z}{}^{(3)}(x)$ be the contribution of primes to $\mathfrak{S}_{v,z}(x)$ satisfying R1 and R2, but not satisfying R3. Note that if R1, R2 are satisfied and R3 is not satisfied, then there are at least two distinct primes p_i, p_j such that $q|p_i-1$ and $q|p_j-1$. Let $\mathbf{u}_{q,q} := (q,q,1,\dots,1)$. Suppose first that this happens with $q>z^4$. Then by (19), the contribution is

$$\ll \sum_{q>z^4}\binom{v}{2}\mathfrak{S}_{\mathbf{u}_{q,q},v,z}(x) \ll \sum_{q>z^4}\frac{v^2}{\phi(q)^2}\mathfrak{S}_{v,z}(x)\log^2 z \ll \frac{v^2\log z}{z^4}\mathfrak{S}_{v,z}(x).$$

Suppose that this happens with $z^2<q\le z^4$. Then by (20), the contribution is

$$\ll \sum_{z^2<q\le z^4}\binom{v}{2}\mathfrak{S}_{\mathbf{u}_{q,q},v,z}(x) \ll \sum_{z^2<q\le z^4}\frac{v^2}{q^2}\mathfrak{S}_{v,z}(x) \ll \frac{v^2}{z^2\log z}\mathfrak{S}_{v,z}(x).$$

Thus, we have

$$\mathfrak{S}_{v,z}{}^{(3)}(x) \ll \frac{v^2}{z^2\log z}\mathfrak{S}_{v,z}(x).$$

We write $\mathfrak{S}_{v,z}{}^{(0)}(x)$ to denote the contribution of those primes to $\mathfrak{S}_{v,z}(x)$ satisfying all three restrictions R1, R2, and R3. By the above estimates, we have

$$\begin{aligned}
\mathfrak{S}_{v,z}{}^{(0)}(x) &\ge \mathfrak{S}_{v,z}(x) - \mathfrak{S}_{v,z}{}^{(1)}(x) - \mathfrak{S}_{v,z}{}^{(2)}(x) - \mathfrak{S}_{v,z}{}^{(3)}(x)\\
&= \mathfrak{S}_{v,z}(x)\left(1+O\left(\frac{\log^3 z}{z}\right)+O\left(\frac{v}{z\log z}\right)+O\left(\frac{v^2}{z^2\log z}\right)\right).
\end{aligned}$$

Therefore,

$$\mathfrak{S}_{v,z}{}^{(0)}(x) = \mathfrak{S}_{v,z}(x)\left(1+O\left(\frac{\log^3 z}{z}\right)+O\left(\frac{v}{z\log z}\right)+O\left(\frac{v^2}{z^2\log z}\right)\right). \tag{23}$$

3. Proof of Theorem 1.1

We set

$$v = v(x) := \left\lfloor c\sqrt{\frac{\log x}{\log_2 x}} \right\rfloor, \quad z = z(x) := \sqrt{\log x},$$
$$y := \exp\left(\sqrt{\log x}\right)$$

with a positive constant c to be determined.

Consider a subset $Q_z(x)$ of primes defined by:

$$Q = Q_z(x) := \{p : p \le x, \ q^2 \nmid p-1 \ \text{ for any prime } q > z\}.$$

We define $\mathcal{N}$, $\mathcal{M}$ by:

$$\mathcal{N} = \mathcal{N}_v(x) := \{n \le x : n \ \text{ is square-free, } p|n \ \Rightarrow p \in Q, \ w(n) = v\},$$
$$\mathcal{M} = \mathcal{M}_v(x) := \{n \le x : n \in \mathcal{N}, \ q^2 \nmid \phi(n) \ \text{ for any prime } q > z^2\}.$$

We write

$$V_{\mathcal{M}}(x) := \sum_{n \in \mathcal{M}} \frac{\tau_z(\lambda(n))}{n}, \quad \tau_z''(n) := \prod_{p|n} \tau_z(p-1).$$

We also write

$$W_{\mathcal{M}} := \sum_{n \in \mathcal{M}} \frac{\tau_z''(n)}{n}, \quad W_{\mathcal{M}}' := \sum_{n \in \mathcal{M}} \frac{\tau_{z^2}''(n)}{n}.$$

By (23), the contribution of those primes satisfying R1, R2, and R3 to $\mathfrak{S}_{v,z}(x)$, which we wrote as ${\mathfrak{S}_{v,z}}^{(0)}(x)$ satisfies

$$\begin{aligned}{\mathfrak{S}_{v,z}}^{(0)}(x) &= \mathfrak{S}_{v,z}(x)\left(1 + O\left(\frac{\log^3 z}{z}\right) + O\left(\frac{v}{z\log z}\right) + O\left(\frac{v^2}{z^2\log z}\right)\right). \\ &= \mathfrak{S}_{v,z}(x)\left(1 + O\left(\frac{1}{\log_2 x}\right)\right).\end{aligned}$$

Then by Lemma 2.6(18) and Stirling's formula,

$$W_{\mathcal{M}} \ge \frac{1}{v!}{\mathfrak{S}_{v,z}}^{(0)}(x) \asymp \frac{1}{v}\left(\frac{e}{v}\right)^{2v}\left(c_1\frac{\log x}{\log z}\right)^v (1+o(1))^v$$

Thus,

$$W_{\mathcal{M}} \gg \exp\left(\sqrt{\frac{\log x}{\log_2 x}}\,(2c + c\log c_1 - 2c\log c + c\log 2 + o(1))\right).$$

Maximizing $2c + c\log c_1 - 2c\log c + c\log 2$ by the first derivative, we have $c = \sqrt{2}e^{-\gamma/2}$, hence

$$W_{\mathcal{M}} \gg \exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right).$$

For $W_{\mathcal{M}}'$, we have by (23), the contribution of those primes satisfying R1, R2, and R3 to $\mathfrak{S}_{v,z^2}(x)$, say ${\mathfrak{S}_{v,z^2}}^{(0')}(x)$ satisfies

$$\begin{aligned}{\mathfrak{S}_{v,z^2}}^{(0')}(x) &= \mathfrak{S}_{v,z^2}(x)\left(1 + O\left(\frac{\log^3 z}{z^2}\right) + O\left(\frac{v}{z\log z}\right) + O\left(\frac{v^2}{z^2\log z}\right)\right). \\ &= \mathfrak{S}_{v,z^2}(x)\left(1 + O\left(\frac{1}{\log_2 x}\right)\right).\end{aligned}$$

Then by Lemma 2.6(18) and Stirling's formula, as $x \to \infty$,

$$W_{\mathcal{M}}' \ge \frac{1}{v!}{\mathfrak{S}_{v,z^2}}^{(0')}(x) \asymp \frac{1}{v}\left(\frac{e}{v}\right)^{2v}\left(c_1\frac{\log x}{\log z^2}\right)^v (1+o(1))^v$$

Thus,

$$W'_{\mathcal{M}} \gg \exp\left(\sqrt{\frac{\log x}{\log_2 x}}\,(2c + c\log c_1 - 2c\log c + o(1))\right).$$

Maximizing $2c + c\log c_1 - 2c\log c$ by the first derivative, we have $c = e^{-\gamma/2}$, hence as $x \to \infty$,

$$W'_{\mathcal{M}} \gg \exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1 + o(1))\right).$$

Therefore, we have just proved the lower bounds of the following:

Theorem 3.1. *For $z = \sqrt{\log x}$, as $x \to \infty$,*

$$\sum_{n \le x} \mu^2(n)\frac{\tau''_z(n)}{n} = \exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1 + o(1))\right), \tag{24}$$

and

$$\sum_{n \le x} \mu^2(n)\frac{\tau''_{z^2}(n)}{n} = \exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1 + o(1))\right). \tag{25}$$

Note that the upper bounds follow from Rankin's method as in [LP, Theorem 1].

We proceed the similar argument as in [LP]. Let $\mathcal{M} = \mathcal{M}_v(x)$ be as above with the choice $c = e^{-\gamma/2}$. Now, for $n \in \mathcal{M}$, we have

$$\tau_z(\phi(n)) = \tau_{z,z^2}(\phi(n))\tau_{z^2}(\phi(n)) \ge \tau_{z^2}(\phi(n)) = \tau''_{z^2}(n),$$
$$\tau_z(\lambda(n)) = \tau_{z,z^2}(\lambda(n))\tau_{z^2}(\lambda(n)) \ge \tau_{z^2}(\lambda(n)) = \tau''_{z^2}(n).$$

Then as $x \to \infty$,

$$V_{\mathcal{M}}(x) \ge W'_{\mathcal{M}} \gg \exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1 + o(1))\right).$$

The argument proceeds as in [LP]. Let $\mathcal{M}'$ be defined by

$$\mathcal{M}' := \left\{np : n \in \mathcal{M}_v(xy^{-1}),\quad p \text{ is a prime},\quad p \le \frac{x}{n}\right\}.$$

For those $n' = np \in \mathcal{M}'$, we have

$$\tau(\lambda(np)) \ge \tau(\lambda(n)) \ge \tau_z(\lambda(n)),$$

and a given $n' \in \mathcal{M}'$ has at most $v + 1$ decompositions of the form $n' = np$ with $n \in \mathcal{M}_v(xy^{-1})$, $p \le \frac{x}{n}$.

Since $n \le xy^{-1}$ for $n \in \mathcal{M}_v(xy^{-1})$, the number of p in $p \le \frac{x}{n}$ is

$$\pi\left(\frac{x}{n}\right) \gg \frac{x}{n\log x}.$$

Note that $\log y = \sqrt{\log x} = o(\log x)$. This gives

$$V_{\mathcal{M}}(xy^{-1}) \gg \exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1 + o(1))\right).$$

Then

$$\sum_{n \le x} \tau(\lambda(n)) \ge \sum_{n \in \mathcal{M}'} \tau(\lambda(n)) \gg V_{\mathcal{M}}(xy^{-1})\frac{x}{v\log x} \gg x\exp\left(2e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1 + o(1))\right).$$

This completes the proof of Theorem 1.1.

Remarks.

1. In the proof of Theorem 1.1, we dropped $\tau_{z,z^2}(\phi(n))$. This is where a prime $z < q \le z^2$ can divide multiple $p_i - 1$ for $i = 1, 2, \cdots, v$, and that is the main difficulty in obtaining more precise formulas for $\sum_{n\le x}\tau(\phi(n))$ and $\sum_{n\le x}\tau(\lambda(n))$.

2. We will see a heuristic argument suggesting that as $x \to \infty$,

$$\sum_{n\le x}\tau(\lambda(n)) = x\exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right),$$

and hence,

$$\sum_{n\le x}\tau(\phi(n)) = x\exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right).$$

However, we have

$$\sum_{n\le x}\tau(\lambda(n)) = o\left(\sum_{n\le x}\tau(\phi(n))\right).$$

We will prove this in the following section. The prime 2 plays a crucial role in the proof of Theorem 1.2.

4. Proof of Theorem 1.2

We put k and w as in [LP]:

$$k = \lfloor A\log_2 x\rfloor,\quad \omega = \left\lfloor \frac{\sqrt{\log x}}{\log_2^2 x}\right\rfloor.$$

Here, A is a positive constant to be determined. Also, define $\mathcal{E}_1(x)$, $\mathcal{E}_2(x)$ and $\mathcal{E}_3(x)$ in the same way:

$$\mathcal{E}_1(x) := \{n \le x : 2^k | n \text{ or there is a prime } p|n \text{ with } p \equiv 1 \bmod 2^k\},$$

$$\mathcal{E}_2(x) := \{n \le x : \omega(n) \le \omega\},$$

and

$$\mathcal{E}_3(x) := \{n \le x\} - (\mathcal{E}_1(x) \cup \mathcal{E}_2(x)).$$

We need the following lemma.

Lemma 4.1. *For any $2 \le y \le x$, we have*

$$\sum_{n\le \frac{x}{y}}\frac{\tau(\phi(n))}{n} \ll \frac{\log^5 x}{x}\sum_{n\le x}\tau(\phi(n)).$$

Proof. As in the proof of [LP, Theorem 1], we use the square-free kernel $k = k(n)$ (if a prime p divides n, then $p|k$, and k is a square-free positive integer which divides n) and the factorization $n = mk$ to rewrite the sum as

$$\sum_{n\le \frac{x}{y}}\frac{\tau(\phi(n))}{n} \le \sum_{k\le\frac{x}{y}}\mu^2(k)\sum_{m\le\frac{x}{ky}}\frac{\tau(m)\tau(\phi(k))}{mk}$$
$$\ll \sum_{k\le\frac{x}{y}}\mu^2(k)\frac{\tau(\phi(k))}{k}\log^2 x.$$

Note that we have uniformly $w(n) \ll \log x$. Find v such that

$$\sum_{\substack{k\le\frac{x}{y}\\ \omega(k)=v}}\mu^2(k)\frac{\tau(\phi(k))}{k}$$

is maximal. Then we have

$$\sum_{k\leq \frac{x}{y}} \mu^2(k)\frac{\tau(\phi(k))}{k} \ll \log x \sum_{\substack{k\leq \frac{x}{y}\\ \omega(k)=v}} \mu^2(k)\frac{\tau(\phi(k))}{k}.$$

We adopt an idea from the proof of Theorem 1.1. Let $\mathcal{M}=\mathcal{M}_v(xy^{-1})$ be the set of square-free numbers $k\leq xy^{-1}$ with $\omega(k)=v$. Define

$$\mathcal{M}' := \left\{kp : k\in\mathcal{M}_v(xy^{-1}),\quad p \text{ is a prime},\quad p\leq \frac{x}{k}\right\}.$$

For those $n'=kp\in\mathcal{M}'$ with $k\in\mathcal{M}$, we have

$$\tau(\phi(kp))\geq \tau(\phi(k)),$$

and any given $n'\in\mathcal{M}'$ has at most $v+1$ decompositions of the form $n'=kp$ with $k\in\mathcal{M}$, $p\leq\frac{x}{k}$.

Since the number of p satisfying $p\leq\frac{x}{k}$ is

$$\pi\left(\frac{x}{k}\right)\gg \frac{x}{k\log x},$$

it follows that

$$\sum_{n\leq x}\tau(\phi(n))\geq \sum_{n\in\mathcal{M}'}\tau(\phi(n))\gg \sum_{\substack{k\leq \frac{x}{y}\\ \omega(k)=v}} \mu^2(k)\frac{\tau(\phi(k))}{k}\frac{x}{v\log x}.$$

Since $v\ll\log x$, we have

$$\sum_{\substack{k\leq \frac{x}{y}\\ w(k)=v}} \mu^2(k)\frac{\tau(\phi(k))}{k}\ll \frac{\log^2 x}{x}\sum_{n\leq x}\tau(\phi(n)).$$

This gives

$$\sum_{k\leq \frac{x}{y}} \mu^2(k)\frac{\tau(\phi(k))}{k}\ll \frac{\log^3 x}{x}\sum_{n\leq x}\tau(\phi(n)).$$

Then the result follows. □

For $n\in\mathcal{E}_1(x)$, we have by Lemma 2.3 and Lemma 4.1,

$$\begin{aligned}\sum_{n\in\mathcal{E}_1(x)}\tau(\lambda(n)) &\leq x\sum_{n\in\mathcal{E}_1(x)}\frac{\tau(\phi(n))}{n}\\ &\leq x\frac{\tau(2^k)}{2^k}\sum_{m\leq\frac{x}{2^k}}\frac{\tau(\phi(m))}{m} + x\sum_{\substack{p\leq x\\ p\equiv 1 \bmod 2^k}}\frac{\tau(p-1)}{p}\sum_{m\leq\frac{x}{p}}\frac{\tau(\phi(m))}{m}\\ &\ll \log^5 x\left(\frac{\tau(2^k)}{\phi(2^k)}\log x\sum_{n\leq x}\tau(\phi(n))\right)\\ &\ll \log^6 x\frac{A\log_2 x}{\log^{A\log 2} x}\sum_{n\leq x}\tau(\phi(n)).\end{aligned}$$

If we take $A\log 2>7$, then we obtain that

$$\sum_{n\in\mathcal{E}_1(x)}\tau(\lambda(n)) = o\left(\sum_{n\leq x}\tau(\phi(n))\right).$$

For $n\in\mathcal{E}_2(x)$, we use the square-free kernel $k=k(n)$ and the factorization $n=mk$ as before,

$$\begin{aligned}\sum_{n\in\mathcal{E}_2(x)}\tau(\lambda(n)) &\le \sum_{n\in\mathcal{E}_2(x)}\tau(\phi(n))\\ &\ll \sum_{\substack{k\le x\\ \omega(k)\le\omega}}\mu^2(k)\sum_{m\le\frac{x}{k}}\tau(m)\tau(\phi(k))\\ &\ll \sum_{\substack{k\le x\\ \omega(k)\le\omega}}\mu^2(k)\frac{x}{k}(\log x)\tau(\phi(k))\\ &\ll x\omega\log x\left(\sum_{p\le x}\frac{\tau(p-1)}{p}\right)^{\omega}\\ &\ll x(\log x)^{\frac{3}{2}}(C\log x)^{\omega}\ll x\exp\left(2\frac{\sqrt{\log x}}{\log_2 x}\right).\end{aligned}$$

Thus, by Theorem 1.1,

$$\sum_{n\in\mathcal{E}_2(x)}\tau(\lambda(n))=o\left(\sum_{n\le x}\tau(\phi(n))\right).$$

For $n\in\mathcal{E}_3(x)$, we follow the method of [LP]. We have

$$\frac{\tau(\phi(n))}{\tau(\lambda(n))}\ge\frac{\omega}{k}\gg\frac{\sqrt{\log x}}{\log_2^3 x}.$$

Then

$$\sum_{n\in\mathcal{E}_3(x)}\tau(\lambda(n))\ll\frac{\log_2^3 x}{\sqrt{\log x}}\sum_{n\in\mathcal{E}_3(x)}\tau(\phi(n))\le\frac{\log_2^3 x}{\sqrt{\log x}}\sum_{n\le x}\tau(\phi(n)).$$

Therefore, putting these together, we have

$$\sum_{n\le x}\tau(\lambda(n))\ll\frac{\log_2^3 x}{\sqrt{\log x}}\sum_{n\le x}\tau(\phi(n)),$$

and Theorem 1.2 follows.

5. Heuristics

Recall that $\tau_z(\lambda(n))=\tau_{z,z^2}(\lambda(n))\tau_{z^2}(\lambda(n))$. Let $\mathcal{M}$ be the set defined in Section 3 with the choice of $v=\left\lfloor\sqrt{2}e^{-\gamma/2}\sqrt{\frac{\log x}{\log_2 x}}\right\rfloor$. As in Section 3, we have $\tau_{z^2}(\lambda(n))=\tau''_{z^2}(n)$ for $n\in\mathcal{M}$. It is important to note that $q^2\nmid p_i-1$ for any primes $p_i|n$ and $q>z$. Also, we have $q^2\nmid\phi(n)$ for $q>z^2$. Thus, it is enough to focus on the sum $V_{\mathcal{M}}(x)$. If we could prove that $V_{\mathcal{M}}(x)=\sum_{n\in\mathcal{M}}\frac{\tau_z(\lambda(n))}{n}\gg\exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right)$, then the same argument as in Theorem 1.1 would allow $\sum_{n\le x}\tau(\lambda(n))\gg x\exp\left(2\sqrt{2}e^{-\frac{\gamma}{2}}\sqrt{\frac{\log x}{\log_2 x}}(1+o(1))\right)$. We need the contribution of $\tau_{z,z^2}(\lambda(n))$ over $n\in\mathcal{M}$. Let $\mathfrak{S}_{v,z}(x)$ be the sum defined in Section 2, and define

$$\mathfrak{U}_{v,z}(x):=\sum_{\left(\frac{\log p_1}{\log x},\dots,\frac{\log p_v}{\log x}\right)\in\cup M_v}\frac{\tau_{z,z^2}(\mathrm{lcm}(p_1-1,p_2-1,\dots,p_v-1))}{\tau_{z,z^2}(p_1-1)\tau_{z,z^2}(p_2-1)\cdots\tau_{z,z^2}(p_v-1)}\frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v}.$$

We have also defined in Section 2 that for $\mathbf{u}=(u_1,\dots,u_v)$ with $1\le u_i\le x$,

$$\mathfrak{S}_{\mathbf{u},v,z}(x):=\sum_{\substack{\left(\frac{\log p_1}{\log x},\dots,\frac{\log p_v}{\log x}\right)\in\cup M_v\\ \forall_i,\ p_i\equiv 1\bmod u_i}}\frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v},$$

We need to extend Lemma 2.6 to cover all components of $\mathbf{u}$.

Lemma 5.1. *Let $\log^{\frac{1}{A}} x < z \le \log^A x$, then for $\mathbf{u} = (u_1, u_2, \ldots, u_v)$ with $1 \le u_i \le x$,*

$$\mathfrak{S}_{\mathbf{u},v,z}(x) \ll \frac{\tau(u_1)\tau(u_2)\cdots\tau(u_v)}{\phi(u_1)\phi(u_2)\cdots\phi(u_v)}\mathfrak{S}_{v,z}(x)(1+o(1))^k \log^k z, \tag{26}$$

where $0 \le k \le v$ is the number of u_i's that are not 1.

Assume that each u_i, $1 \le i \le v$ is either 1 or a positive integer with $p(u_i) > z$, $u_i < (\log x)^{A_1}$ and $\tau(u_i) < A_1$. Then

$$\mathfrak{S}_{\mathbf{u},v,z}(x) = \frac{\tau(u_1)\tau(u_2)\cdots\tau(u_v)}{u_1u_2\cdots u_v}\mathfrak{S}_{v,z}(x)\left(1+o(1)\right)^k, \tag{27}$$

where $0 \le k \le v$ is the number of u_i's that are not 1.

The same proof as in Lemma 2.6 applies with the need of considering all components of $\mathbf{u}$.

Fix a prime $z < q \le z^2$. Consider the number X_q of primes $p_1, \ldots, p_v$ such that q divides $p_i - 1$. By Lemma 5.1, it is natural to model X_q by a binomial distribution with parameters v and $\frac{2}{q}$. In fact, Lemma 5.1 implies that

Lemma 5.2. *For any $0 \le k \le v$, as $x \to \infty$,*

$$\begin{aligned} P(X_q = k) :&= \frac{1}{\mathfrak{S}_{v,z}(x)} \sum_{\substack{\left(\frac{\log p_1}{\log x},\ldots,\frac{\log p_v}{\log x}\right)\in\cup M_v \\ \text{Exactly } k \text{ primes } p_i \text{ satisfy } q|p_i-1}} \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v} \\ &= \binom{v}{k}\left(\frac{2}{q}\right)^k\left(1-\frac{2}{q}\right)^{v-k}(1+o(1))^v. \end{aligned}$$

Here, the functions implied in $1+o(1)$ only depend on x and do not depend on k.

Denote by A_q the contribution of a power of q in

$$\frac{\tau_{z,z^2}(\mathrm{lcm}(p_1-1,p_2-1,\ldots,p_v-1))}{\tau_{z,z^2}(p_1-1)\tau_{z,z^2}(p_2-1)\cdots\tau_{z,z^2}(p_v-1)}.$$

Similarly, denote by $A_{q_1,\cdots,q_j}$ the contribution of powers of $q_1,\cdots,q_j$ in the above. Let

$$B_{z,v} := \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v}.$$

We can combine the contributions of finite number of primes $q_1, \ldots, q_j$ in $(z, z^2]$. For these multiple primes, Lemma 5.2 becomes

Lemma 5.3. *For any $0 \le k_1, \ldots, k_j \le v$, as $x \to \infty$,*

$$\begin{aligned} P(X_{q_1} = k_1, \ldots, X_{q_j} = k_j) :&= \frac{1}{\mathfrak{S}_{v,z}(x)} \sum_{\substack{\left(\frac{\log p_1}{\log x},\ldots,\frac{\log p_v}{\log x}\right)\in\cup M_v \\ \text{For each } s=1,\ldots,j, \\ \text{exactly } k_s \text{ primes } p_i \text{ satisfy } q_s|p_i-1}} \frac{\tau_z(p_1-1)\tau_z(p_2-1)\cdots\tau_z(p_v-1)}{p_1p_2\cdots p_v} \\ &= \prod_{s\le j}\binom{v}{k_s}\left(\frac{2}{q_s}\right)^{k_s}\left(1-\frac{2}{q_s}\right)^{v-k_s}(1+o(1))^v. \end{aligned}$$

Here, the functions implied in $1+o(1)$ only depend on j, x and they do not depend on k_s.

This shows that the random variables X_{q_i} behave similar as independent binomial distributions. For $z < q \le z^2$, we have $A_q = \frac{2}{2^k}$ for $k \ge 1$, and $A_q = 1$ for $k = 0$. Thus, the contribution of this prime q is

$$\mathbf{E}[A_q] = \left(2\left(1-\frac{1}{q}\right)^v - \left(1-\frac{2}{q}\right)^v\right)(1+o(1))^v.$$

For distinct primes $q_1, \dots, q_j$ in $(z, z^2]$, the contribution of these primes is

$$\mathbf{E}[A_{q_1,\dots,q_j}] = \prod_{s \le j} \left(2\left(1 - \frac{1}{q_s}\right)^v - \left(1 - \frac{2}{q_s}\right)^v \right) (1 + o(1))^v,$$

where the function implied in $1 + o(1)$ only depends on j, x.

Then, we conjecture that the contribution of all primes in $z < q \le z^2$ will be

Conjecture 5.1. *As $x \to \infty$, we have*

$$\mathfrak{U}_{v,z}(x) = \prod_{z < q \le z^2} \left(2\left(1 - \frac{1}{q}\right)^v - \left(1 - \frac{2}{q}\right)^v \right) \mathfrak{S}_{v,z}(x)(1 + o(1))^v.$$

It is clear that

$$2\left(1 - \frac{1}{q}\right)^v - \left(1 - \frac{2}{q}\right)^v = 1 + o\left(\frac{v}{q}\right).$$

Thus, we have as $x \to \infty$,

$$\prod_{z < q \le z^2} \left(2\left(1 - \frac{1}{q}\right)^v - \left(1 - \frac{2}{q}\right)^v \right) = (1 + o(1))^v.$$

Therefore, we obtain the following heuristic result according to Conjecture 5.1.

Conjecture 5.2. *As $x \to \infty$, we have*

$$\mathfrak{U}_{v,z}(x) = \mathfrak{S}_{v,z}(x)(1 + o(1))^v.$$

Then Conjecture 1.1 follows from Lemma 2.6.

Remarks.

We were unable to prove Conjecture 1.1. The main difficulty is due to the short range of u in Corollary 2.1. Because of the range of u, we could not extend Lemma 5.3 to all primes in $z < q \le z^2$.

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