TWO REMARKS ON THE LARGEST PRIME FACTORS OF n AND $n+1$

SUNGJIN KIM

Abstract. Let $P(n)$ be the largest prime factor of n. We give an alternative proof of the existence of infinitely many n such that $P(n) > P(n+1) > P(n+2)$. Further, we prove that the set $\{P(n+1)/P(n)\}_{n\in\mathbb{N}}$ has infinitely many limit points $\{0, x_n, 1, y_n\}_{n\in\mathbb{N}}$ with $0 < x_n < 1 < y_n$ and $\lim x_n = \lim y_n = 1$.

1. Introduction

Let $n \geq 2$ be a positive integer. Denote by $P(n)$ the largest prime factor of n. Erdős and Pomerance [EP] proved that the number of $n \leq x$ such that $P(n) < P(n+1)$ is at least $0.0099x$, and the same holds for $P(n) > P(n+1)$. This lower density 0.0099 is subsequently improved by several authors (0.05544 by de la Bretèche, Pomerance and Tenenbaum [BPT], 0.1063 by Z. Wang [W]). The current record holders are Lü and Wang [LW], who proved that the lower density is at least 0.2017. Erdős and Pomerance [EP] also note that the three patterns $P(n) < P(n+1) > P(n+2)$, $P(n) > P(n+1) < P(n+2)$, and $P(n) < P(n+1) < P(n+2)$ occur infinitely often. They presented a simple proof for the infinitude of the third pattern. Namely, they take

$$n = p^{2^m} - 1,\ n+1 = p^{2^m},\ \text{and } n+2 = p^{2^m} + 1$$

where p is prime and $m = \inf\{k | P(p^{2^k} + 1) > p\}$. They left the infinitude of the fourth pattern $P(n) > P(n+1) > P(n+2)$ as a question. This question was later solved by Balog [B], who showed that the number of occurrence of this pattern for $n \leq x$ is $\gg \sqrt{x}$. Building on earlier results by Matomäki, Radziwiłł, and T. Tao [MRT], and Teräväinen [Te], Tao and Teräväinen [TT] proved that the following sets have positive lower density: $\{n \in \mathbb{N} \mid P(n) < P(n+1) < P(n+2) > P(n+3)\}$ and $\{n \in \mathbb{N} \mid P(n) > P(n+1) > P(n+2) < P(n+3)$. Using the Maynard-Tao theorem [Pb], we provide a simple alternative proof of the infinitude of the patterns $P(n) < P(n+1) < P(n+2)$ and $P(n) > P(n+1) > P(n+2)$. We prove that both patterns occur for $\gg x/(\log x)^{50}$ values of $n \leq x$. The result is weaker than Tao and Teräväinen, and stronger than Balog.

Theorem 1.1. *For sufficiently large x, we have*

$$\#\{n \leq x \mid P(n) < P(n+1) < P(n+2)\} \gg \frac{x}{(\log x)^{50}} \text{ and}$$

$$\#\{n \leq x \mid P(n) > P(n+1) > P(n+2)\} \gg \frac{x}{(\log x)^{50}}.$$

Erdős and Pomerance [EP, Theorem 1] proved that for any $\epsilon > 0$, there is $\delta > 0$ such that the number of $n \leq x$ with

$$x^{-\delta} < \frac{P(n+1)}{P(n)} < x^{\delta}$$

is less than ϵx. They remarked that this means $P(n)$ and $P(n+1)$ are usually not close. In the opposite direction, we prove that this ratio can approach arbitrarily close to 1 from both sides.

Theorem 1.2. *For any $\epsilon > 0$, we have*

$$\#\left\{n \leq x \,\middle|\, 1 \leq \frac{P(n+1)}{P(n)} < 1 + \epsilon\right\} \gg_{\epsilon} \frac{x}{(\log x)^{50}} \text{ and}$$

$$\#\left\{n \leq x \,\middle|\, 1 - \epsilon < \frac{P(n+1)}{P(n)} \leq 1\right\} \gg_{\epsilon} \frac{x}{(\log x)^{50}}.$$

Let $R := \{\frac{P(n+1)}{P(n)} \mid n \in \mathbb{N}\}$. As a direct consequence of Theorem 1.2, we obtain that $1 \in \overline{R}$. From the proof of Theorem 1.2, we obtain a finite set $\{a_1, \dots, a_{50}\} \subseteq \mathbb{N}$ with $1 \leq a_i < a_j$ for each $1 \leq i < j \leq 50$ such that $a_{j_1}/a_{i_1} \in \overline{R} \cap (1, \infty)$ for some $1 \leq i_1 < j_1 \leq 50$, and $a_{i_2}/a_{j_2} \in \overline{R} \cap (0, 1)$ for some $1 \leq i_2 < j_2 \leq 50$. Changing a_i with ϵ, we obtain a sequence rational numbers with $0 < x_n < 1 < y_n$ such that $\lim x_n = \lim y_n = 1$ and $\{x_n, y_n\}_{n \in \mathbb{N}} \subseteq \overline{R}$.

In fact, there is an elementary proof of $1 \in \overline{R}$. This elementary proof is based on a solution ($\epsilon = 1$ in the following argument) that appeared in Mathematics Stack Exchange [C] by user Barry Cipra. For any $0 < \epsilon \leq 1$, take primes p and q satisfying $p < q < (1+\epsilon)p$ so that

$$\frac{1}{1+\epsilon} < \frac{p}{q} < 1 < \frac{q}{p} < 1 + \epsilon.$$

By Bézout's identity, there are integers u and v with $0 < u < q$, $0 < v < p$ and $pu - qv = 1$. Let $U = q - u$, $V = p - v$. Then $qV - pU = 1$. The integers qu and qV have q as the largest prime factor. As $u + U = q < (1+\epsilon)p$, at least one of $u \leq p$ or $U \leq p$ is true. If $u \leq p$ is true, then p is the largest prime factor of pu. If $U \leq p$ is true, then also p is the largest prime factor of pU. Thus, either one of the following is true:

$$n = qv,\ n+1 = pu,\ \frac{P(n+1)}{P(n)} = \frac{p}{q},$$

or

$$n = pU,\ n+1 = qV,\ \frac{P(n+1)}{P(n)} = \frac{q}{p}.$$

Therefore, $1 \in \overline{R}$ follows. From this argument the number of $n \leq x$ with $\frac{1}{1+\epsilon} < \frac{P(n+1)}{P(n)} < 1 + \epsilon$ is $\gg_\epsilon x/(\log x)^2$. Slightly modifying this argument, we have for any $x \in [1, 2]$, either x or $1/x$ is in $\overline{R}$. However, this argument does not determine whether a limit point is in $[0, 1)$ or $(1, \infty)$.

By Dirichlet's theorem on primes in arithmetic progressions, it is easy to see that 0 is also a limit point of R. For if we take a prime $n = ar - 1$, with a large, then $P(n) = ar - 1$ and $P(n+1) \leq \max(a, r)$. Assuming the Prime k-tuples conjecture (Conjecture 2.1), we prove that all nonnegative real numbers are limit points of R.

Theorem 1.3 (Conditional)**.** *Assuming the Prime k-tuples conjecture (Conjecture 2.1), we have $\overline{R} = [0, \infty)$.*

2. Estimates on the Numbers of Prime k-Tuples

A set of k-tuple of linear forms $\{a_1x + b_1, \dots a_kx + b_k\}$ is said to be *admissible* if for any prime p there is $x_p \in \mathbb{Z}$ such that $p \nmid \prod_{i=1}^k (a_i x_p + b_i)$. We consider the tuples with

$$\prod_i a_i \neq 0 \quad \text{and} \quad \prod_{i<j} (a_i b_j - a_j b_i) \neq 0.$$

The following is a special case of Bateman-Horn conjecture (a quantitative estimate on Dickson's Prime k-tuples conjecture).

Conjecture 2.1 (Bateman-Horn)**.** *Let $k \geq 2$ and $A_k = \{a_1x + b_1, \dots, a_kx + b_k\}$ be an admissible set of linear forms. Then for sufficiently large x, the number $R_k(x)$ of $r \leq x$ such that $a_ix + b_i$, $1 \leq i \leq k$ are all prime satisfies*

$$R_k(x) \gg_{A_k} \frac{x}{(\log x)^k}.$$

Substantial progress toward this conjecture begin with Zhang's result [Z] on bounded gaps in primes. Subsequently, Maynard [M] and Polymath8b ([Pb] led by Tao) improved upon Zhang's result. We state a quantitative form of the Maynard-Tao theorem for admissible sets of linear forms. The proof requires slight modifications of [Pb] and the stated lower bound can be found in [Pb, Remark 32]. Note that the following is unconditional.

Lemma 2.1 (Maynard-Tao-Polymath8b)**.** *Let $A = \{a_1 r + b_1, \ldots, a_{50} r + b_{50}\}$ be an admissible set of linear forms. Then for sufficiently large x, the number $R(A, x)$ of $r \le (x - \max_i b_i)/\max_i a_i$ such that at least two of the linear forms are primes satisfies*

$$R(A, x) \gg_A \frac{x}{(\log x)^{50}}.$$

We will apply the above lemma in the following two special cases.

Case 1. $0 < a_1 < \cdots < a_k$ and $b_i = 1$ for all $i = 1, \ldots, k$.

Case 2. $0 < a_1 < \cdots < a_k$ and $b_i = -1$ for all $i = 1, \ldots, k$.

The set of linear forms in these cases is always admissible.

3. The Main Lemma

We construct a special sequence $\{a_i\}$ by the following inductive process.

Lemma 3.1 (The Main Lemma)**.** *Let $k \ge 2$ and $e_k = 1$. For each $0 \le j \le k-2$, assume that $\{e_{k-j}, \ldots, e_k\}$ satisfies*

$$\sum_{s<i\le t} e_i \;\Bigg|\; \sum_{k-j\le i\le s} e_i \quad \text{for any} \quad k-j \le s < t \le k.$$

Let e_{k-j-1} be a multiple of

$$\mathrm{LCM}\left\{ \sum_{s\le i\le t} e_i \;\middle|\; k-j \le s < t \le k \right\}.$$

Then $a_i = \sum_{m\le i} e_m$ satisfies $0 < a_j - a_i \mid a_i$ for each $1 \le i < j \le k$.

Proof. The proof is clear from the inductive construction. □

We exhibit some sequences $\{a_i\}$ that can be produced by the Main Lemma.

Examples. If $k = 2$, then let $\{e_1, e_2\} = \{1, 1\}$ and $\{a_1, a_2\} = \{1, 2\}$.

If $k = 3$, then let $\{e_1, e_2, e_3\} = \{2, 1, 1\}$ and $\{a_1, a_2, a_3\} = \{2, 3, 4\}$.

If $k = 4$, then let $\{e_1, e_2, e_3, e_4\} = \{12, 2, 1, 1\}$ and $\{a_1, a_2, a_3, a_4\} = \{12, 14, 15, 16\}$.

If $k = 5$, then let $\{e_1, e_2, e_3, e_4, e_5\} = \{1680, 12, 2, 1, 1\}$ and

$\{a_1, a_2, a_3, a_4, a_5\} = \{1680, 1692, 1694, 1695, 1696\}$.

Note that e_1 can be made arbitrarily large in the final inductive step. We will use the sequence $\{a_i\}_{1\le i\le 50}$.

Lemma 3.2. *Let $\{a_i\}_{1\le i\le 50}$ be a sequence produced in the Main Lemma. That is, $0 < a_j - a_i \mid a_i$ for each $1 \le i < j \le 50$. Suppose that $a_i r + 1$ and $a_j r + 1$ are primes. Then by taking*

$$n = \frac{a_i}{a_j - a_i}(a_j r + 1), \;\; n + 1 = \frac{a_j}{a_j - a_i}(a_i r + 1), \tag{1}$$

we have for sufficiently large r,

$$P(n) = a_j r + 1, \;\; P(n+1) = a_i r + 1.$$

Suppose now that $a_i r - 1$ and $a_j r - 1$ are primes. Then by taking

$$n = \frac{a_j}{a_j - a_i}(a_i r - 1), \;\; n + 1 = \frac{a_i}{a_j - a_i}(a_j r - 1), \tag{2}$$

we have for sufficiently large r,

$$P(n) = a_i r - 1,\ \ P(n+1) = a_j r - 1.$$

Proof. We take large enough r so that $a_1 r - 1$ exceeds the largest prime factor of $\prod a_i$. □

Remark. The author recently learned that a sequence $\{a_i\}$ with the property $0 < a_j - a_i \mid a_i$ for each $1 \le i < j$ was obtained earlier by Heath-Brown [Hb, Lemma 1], and such a sequence is used in an unpublished work of Maynard and Ford [F, Theorem 7.18]. Using such a sequence and Lemma 3.2(1), Maynard and Ford proved that there is a constant $B > 0$ so that for infinitely many n, $P(n) \ge n/B$ and $P(n+1) \ge (n+1)/B$.

4. Proof of Theorems

4.1. **Proof of Theorem 1.1.** Let $\{a_i\}_{1\le i\le 50}$ be a sequence produced in the Main Lemma. We apply (1) of Lemma 3.2. By letting $n+2$ divisible by $\frac{a_j}{a_j-a_i}+1$, we obtain $P(n) > P(n+1) > P(n+2)$ for n in (1). For this idea to work, we need to require r to be divisible by $\frac{a_j}{a_j-a_i}+1$ for any choice of $1 \le i < j \le 50$. To see this, we let

$$M = \mathrm{LCM}\left\{ \frac{a_j}{a_j - a_i} + 1 \;\middle|\; 1 \le i < j \le 50 \right\}.$$

Then we work with the admissible set of linear forms $\{a_i M r + 1\}_{1\le i\le 50}$. By Lemma 2.1 and the pigeon-hole principle, there is a pair (i,j), $1 \le i < j \le 50$ depending on x such that $a_i M r + 1$ and $a_j M r + 1$ are primes for $\gg x/(\log x)^{50}$ values of $r \le (x - a_{50})/(a_{50}^2 M)$. For such $r \ge r_0$, we have $n = \frac{a_i}{a_j-a_i}(a_j M r + 1) \le x$, $P(n) = a_j M r + 1$, $P(n+1) = a_i M r + 1$, and $P(n+2) \le (n+2)/(\frac{a_j}{a_j-a_i}+1)$. Thus, $P(n) > P(n+1) > P(n+2)$ is satisfied for such r.

To obtain an analogous result on $P(n) < P(n+1) < P(n+2)$, we apply (2) of Lemma 3.2. By letting $n-1$ divisible by $\frac{a_j}{a_j-a_i}+1$, we obtain $P(n-1) < P(n) < P(n+1)$ for n in (2). Then we work with the admissible set of linear forms $\{a_i M r - 1\}_{1\le i\le 50}$. The rest of the argument is similar to the previous case.

4.2. **Proof of Theorem 1.2.** Let $\epsilon > 0$ be arbitrary. We show that the number of $n \le x$ with $1-\epsilon < \frac{P(n+1)}{P(n)} \le 1$ is $\gg_\epsilon \frac{x}{(\log x)^{50}}$. In the inductive process in Lemma 3.1, we let e_1 be large enough to have

$$1 - \epsilon < \frac{a_i}{a_j} \le 1 \ \text{ for each } 1 \le i < j \le 50.$$

Then we apply Lemma 3.2(1) to conclude the existence of a pair (i,j), $1 \le i < j \le 50$ depending on x such that $a_i r + 1$ and $a_j r + 1$ are primes for $\gg_\epsilon x/(\log x)^{50}$ values of $r \le (x - a_{50})/a_{50}^2$. It is clear that $\frac{a_i r+1}{a_j r+1} = \frac{a_i}{a_j} + \frac{a_j - a_i}{a_j(a_j r+1)}$. Since $P(n) = a_j r + 1$ and $P(n+1) = a_i r + 1$ for such r by Lemma 3.2(1), we have

$$1 \ge \frac{P(n+1)}{P(n)} = \frac{a_i r + 1}{a_j r + 1} > \frac{a_i}{a_j} > 1 - \epsilon.$$

The result now follows.

To obtain an analogous result on $1 \le \frac{P(n+1)}{P(n)} < 1+\epsilon$, we apply Lemma 3.2(2).

4.3. **Proof of Theorem 1.3.** Let a_1 be an even positive integer, and a_2 be a positive integer with $(a_1, a_2) = 1$. By Bezout's identity, we can find positive integers b_1 and b_2 such that $a_1 b_2 - a_2 b_1 = (a_1, a_2) = 1$. The sets of linear forms $\{a_1 r + b_1, a_2 r + b_2\}$ and $\{a_1 r - b_1, a_2 r - b_2\}$ are admissible. By Conjecture 2.1, there are infinitely many r such that both of these forms are primes. We take

$$n = a_2(a_1 r + b_1),\ \ n+1 = a_1(a_2 r + b_2)$$

or

$$n = a_1(a_2 r - b_2),\ \ n+1 = a_2(a_1 r - b_1).$$

If we select r to exceed any prime factor of $a_1 a_2$, then we see in both cases

$$\left\{ \frac{a_1}{a_2}, \frac{a_2}{a_1} \right\} \subseteq \overline{R}.$$

Hence, it follows that any positive rational numbers with numerator and denominator of different parity are limit points of R, and consequently, $\overline{R} = [0, \infty)$.

Acknowledgement. The author thanks Kevin Ford for bringing [Hb], [MRT], and [Te] to his attention. The author also thanks an anonymous referee and Paul Pollack for careful reading of the paper and helpful comments.

References

[B] A. Balog, *On Triplets with Descending Largest Prime Factors*, Studia Scientiarum Mathematicarum Hungaria, 38(2001), pp 45-50.

[BPT] R. de la Bretèche, C. Pomerance, G. Tenenbaum, *Products of Ratios of Consecutive Integers*, The Ramanujan Journal, 9(2005), pp 131-138.

[C] B. Cipra, *Greatest Prime Divisor of Consecutive Integers*, answered and available at `https://math.stackexchange.com/questions/1894873/greatest-prime-divisor-of-consecutive-integers`.

[EP] P. Erdős, C. Pomerance, *On the Largest Prime Factors of* n *and* $n+1$, Aequationes Mathematicae, 17(1978), pp 311-321.

[F] K. Ford, *Sieve Method Lecture Notes, Spring 2020*, available at `https://faculty.math.illinois.edu/~ford/sieve2020.pdf`.

[Hb] D. R. Heath-Brown, *The divisor function at consecutive integers*, Mathematika 31 (1984), no. 1, pp 141149.

[LW] X. Lü, Z. Wang, *On the Largest Prime Factors of Consecutive Integers*, preprint.

[M] J. Maynard, *Small Gaps Between Primes*, Annals of Mathematics, 181(2015), pp 383-413.

[M2] J. Maynard, *Dense Clusters of Primes in Subsets*, Composito Math. 152(2016), pp 1517-1554.

[MRT] K. Matomäki, M. Radziwiłł, T. Tao, *Sign Patterns of the Liouville and Möbius Functions*, Forum Math. Sigma 4(2016), e14, 44 pages.

[Pb] D. H. J. Polymath, *Variants of Selberg Sieve, and Bounded Intervals Containing Many Primes*, Research in Mathematical Sciences, 2014(1), pp 1-83.

[Te] J. Teräväinen, *On Binary Correlations of Multiplicative Functions*, Forum Math. Sigma 6(2018), e10, 41 pages.

[TT] T. Tao, J. Teräväinen, *Value Patterns of Multiplicative Functions and Related Sequences*, Forum of Mathematics, Sigma (2019), e33, 55 pages.

[W] Z. Wang, *On the Largest Prime Factors of Consecutive Integers in Short Intervals*, Proceedings of American Mathematical Society, 145(2017), pp 3211-3220.

[Z] Y. Zhang, *Bounded Gaps between Primes*, Annals of Mathematics, 179(2014), pp 1121-1174.

California State University Northridge, 18111 Nordhoff St, Northridge, CA 91330

Santa Monica College, 1900 Pico Blvd, Santa Monica, CA 90405
E-mail address: `707107@gmail.com`