THE PROBABILITY THAT THE TAYLOR RESOLUTION OF A MONOMIAL IDEAL IS MINIMAL

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Abstract. Let k be an arbitrary field, and let $k[x_1, \ldots, x_n]$ be a polynomial ring on n variables. In this paper we express, in terms of Dedekind numbers, the probability that the Taylor resolution of a squarefree monomial ideal of $k[x_1, \ldots, x_n]$ is minimal. We also show that this probability tends to 0 as n tends to ∞. For $n \leq 9$, we compute the probability that the Taylor resolution is minimal, explicitly.

1. Introduction

The minimal free resolution of a monomial ideal encodes important information about it. Invariants of a monomial ideal such as Betti numbers, regularity, projective dimension, multiplicity, etc., can be read off from its minimal free resolution [Ei,Pe]. Thus, one of the main goals in the area of monomial resolutions is to come up with new constructions that yield the minimal resolutions of infinite families of monomial ideals.

Perhaps, the most important of all these constructions is the Taylor resolution [Ta]. Created by Diana Taylor in 1960, the Taylor resolution produces a free resolution for every single monomial ideal. The problem with this general construction is that it is usually non-minimal. To quantify the adverb "usually", the present article deals with the probability that the Taylor resolution is minimal.

Since the problem of computing the minimal resolution of a monomial ideal can be reduced to the problem of computing the minimal resolution of a squarefree monomial ideal using a technique called polarization [Pe], we will restrict our attention to the latter case. In other words, we will determine how often the Taylor resolution of a squarefree monomial ideal is minimal. To be more precise, if k is a field, and $k[x_1, \ldots, x_n]$ is a polynomial ring on n variables, we will calculate the probability that the Taylor resolution of a squarefree monomial ideal of $k[x_1, \ldots, x_n]$ is minimal. Our computations will be explicit for $n \leq 9$, and will rely on Dedekind numbers for arbitrary values of n. In addition, we will show that the probability that the Taylor resolution is minimal quickly decreases as n increases, and approaches 0 as n tends to ∞.

2. Preliminaries

In this section we set up the necessary background to prove our main results.

Definition 2.1. Let k be a field and M a monomial ideal of the polynomial ring $k[x_1, \ldots, x_n]$, minimally generated by squarefree monomials $m_1, \ldots, m_q$. We will say that M is a **dominant ideal** if, for each i, there is a variable x_{m_i} that appears in the factorization of m_i but not in the factorizations of $m_1, \ldots, \widehat{m_i}, \ldots, m_q$.

Example 2.2. Consider the squarefree monomial ideals $M = (x_1x_4, x_2x_4, x_3x_4)$ and $N = (x_1x_2, x_1x_3, x_2x_4)$ of the polynomial ring $k[x_1, \ldots, x_4]$. Notice that M is dominant because

x_1 appears in the factorization of x_1x_4, but not in the factorizations of x_2x_4 or x_3x_4; likewise, x_2 only appears in the factorization of x_2x_4, and x_3 only appears in the factorization of x_3x_4.

On the other hand, N is not dominant because every variable that appears in the factorization of x_1x_2 also appears in the factorization of some other minimal generator.

Theorem 2.3. *[Al, Theorem 4.4] Let M be a squarefree monomial ideal of $k[x_1, \ldots, x_n]$. Denote by $\mathbb{T}_M$ the Taylor resolution of M. Then $\mathbb{T}_M$ is minimal if and only if M is dominant.*

Proposition 2.4. *Let S_n be the collection of all squarefree monomial ideals of $k[x_1, \ldots, x_n]$. Denote by A_n the dominant ideals of S_n, and by $p(\mathbb{T}_M)$ the probability that the Taylor resolution of an element of S_n is minimal. Then,*

$$p(\mathbb{T}_M) = \frac{|A_n|}{|S_n|}.$$

Proof. By Theorem 2.3, $p(\mathbb{T}_M)$ equals the probability that a randomly chosen element of S_n belongs to A_n; but this probability is $|A_n|/|S_n|$. □

In the next section, we will compute the cardinalities $|A_n|$ and $|S_n|$. (Note: the ideal generated by 1 will be regarded as a squarefree monomial ideal and hence, $(1) \in S_n$.) But what will make the computations possible is the bijective correspondence that we create next.

Suppose that $M = (m_1, \ldots, m_q)$ is a monomial ideal of $k[x_1, \ldots, x_n]$, minimally generated by squarefree monomials $m_1, \ldots, m_q$. It follows that no m_i is divisible by any of $m_1, \ldots, \widehat{m_i}, \ldots, m_q$ (that is, no minimal generator is divisible by another minimal generator). If we denote by $\operatorname{set}(m_i)$ the set of variables that appear in the factorization of m_i, and by $\operatorname{set}(M)$, the collection

$$\operatorname{set}(M) = \{\operatorname{set}(m_1), \ldots, \operatorname{set}(m_q)\},$$

then $\operatorname{set}(M)$ is an antichain (that is, $\operatorname{set}(M)$ is a family of sets where no set is contained in another). Conversely, each antichain different from $\{\}$, defines a squarefree monomial ideal M. Hence, we can establish a bijective correspondence

$$M \longleftrightarrow \operatorname{set}(M)$$

between the squarefree monomial ideals of $k[x_1, \ldots, x_n]$, and the antichains consisting of subsets of $\{x_1, \ldots, x_n\}$, different from $\{\}$. If we call the collection of all nonempty antichains S'_n, then we can say that the correspondence $M \longleftrightarrow \operatorname{set}(M)$ defines a bijection between S_n and S'_n.

The next definition will enable us to establish another important bijective correspondence.

Definition 2.5. Let $\mathscr{F}$ be an antichain consisting of subsets of $\{x_1, \ldots, x_n\}$. We will say that $\mathscr{F}$ is a **dominant antichain** if not set of $\mathscr{F}$ is contained in the union of the others. In other words, $\mathscr{F}$ is dominant if every set of $\mathscr{F}$ contains an element not shared by the other sets of $\mathscr{F}$.

Example 2.6. Consider the antichains $\operatorname{set}(M) = \Big\{\{x_1, x_4\}, \{x_2, x_4\}, \{x_3, x_4\}\Big\}$ and $\operatorname{set}(N) = \Big\{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_4\}\Big\}$ (where M and N are the monomial ideals of Example 2.2), consisting of subsets of $\{x_1, x_2, x_3, x_4\}$. It is easy to verify that $\operatorname{set}(M)$ is a dominant antichain, but $\operatorname{set}(N)$ is not.

It follows from Definitions 2.1 and 2.5, that a squarefree monomial ideal M of $k[x_1,\dots,x_n]$ (or with our prior notation, an element M of S_n) is dominant if and only if the antichain set(M) consisting of subsets of $\{x_1,\dots,x_n\}$ (or with our prior notation, the element set(M) of S'_n) is dominant.

Therefore, if we denote by A'_n the collection of all dominant antichains of S'_n, then the correspondence

$$M \longleftrightarrow \text{set}(M)$$

defines a bijection between the dominant elements of S_n and the dominant elements of S'_n (or with our prior notation, a bijection between A_n and A'_n).

Proposition 2.7. *Let S'_n be the collection of all antichains consisting of subsets of $\{x_1,\dots,x_n\}$ and A'_n the collection of all dominant antichains of S'_n. Denote by $p(\mathbb{T}_M)$ the probability that the Taylor resolution of a squarefree monomial ideal of $k[x_1,\dots,x_n]$ is minimal. Then*

$$p(\mathbb{T}_M) = |A'_n|/|S'_n|.$$

Proof. Given that $|A_n|= |A'_n|$ and $|S_n|= |S'_n|$, the result follows immediately from Proposition 2.4. □

3. Main Results

The numbers $|S'_n|$ are called **Dedekind numbers** and are tabulated at OEIS A014466 [O]. Note that S'_n includes $\{\{\}\}$ but not the empty antichain $\{\}$. No explicit formula is known for $|S'_n|$. There are asymptotic formulas

$$\log_2|S'_n|= \binom{n}{\lfloor n/2\rfloor}\left(1+O\left(\frac{\log n}{n}\right)\right)$$

by D. Kleitman and G. Markowski [KM],

$$|S'_n|\sim\ 2^{\binom{n}{\lfloor n/2\rfloor}} \exp\left(\binom{n}{n/2-1}(2^{-n/2}+n^2 2^{-n-5}-n2^{-n-4})\right) \text{ if } n \text{ is even}$$

and

$$|S'_n|\sim 2^{\binom{n}{\lfloor n/2\rfloor}} \exp\left(\binom{n}{(n-3)/2}\alpha(n)+\binom{n}{(n-1)/2}\beta(n)\right) \text{ if } n \text{ is odd}$$

by A. D. Korshunov [K]. Here,

$$\alpha(n) = 2^{-(n+3)/2} + n^2 2^{-n-6} - n2^{-n-3},$$

$$\beta(n) = 2^{-(n+1)/2} + n^2 2^{-n-4}.$$

A simple but strong lower bound is obtained by counting nonempty families consisting of sets of the same size $\lfloor n/2\rfloor$.

Proposition 3.1. *For $n \geq 1$,*

$$2^{\binom{n}{\lfloor n/2\rfloor}} - 1 \leq |S'_n|.$$

By Inclusion-Exclusion Principle, we obtain a formula for $|A'_n|$.

Theorem 3.2. *The number $|A'_n|$ of dominant antichains of $\{1,\dots,n\}$ is*

$$|A'_n|= \sum_{m=1}^{n}\frac{1}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(2^m-k)^n.$$

Proof. Let $m \geq 1$ and $\mathcal{F} = \{B_1, \dots, B_m\}$. Consider disjoint sets $E_1, \dots, E_m$ defined by

$$E_i = B_i \cap \left(\bigcup_{\substack{1 \leq j \leq m \\ j \neq i}} B_j \right)^c .$$

Consider Venn diagram with m sets. Each of 2^m region in the diagram is labeled by binary digits indicating membership. The sets $E_1, \dots, E_m$ are then labeled by $10 \cdots 0$, $010 \cdots 0$, $\dots$, $0 \cdots 01$ (single 1 in the string of length m), respectively. Note that $\mathcal{F}$ is dominant if and only if $E_i \neq \emptyset$ for each $i = 1, \dots, m$. Assume that each B_i is independently any of 2^n subsets equally likely. This is identified as each element of $\{1, \dots, n\}$ is equally likely to be in any of 2^m regions of Venn diagram. The probability that $E_i \neq \emptyset$ for all i, is obtained by Inclusion-Exclusion Principle,

$$1 - \binom{m}{1} P(E_1 = \emptyset) + \binom{m}{2} P(E_1 = E_2 = \emptyset) - \cdots + (-1)^m \binom{m}{m} P(E_1 = E_2 = \cdots = E_m = \emptyset).$$

The above can be written compactly as

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{(2^m - k)^n}{2^{nm}}.$$

Then the number of m-tuples of sets $(B_1, \dots, B_m)$ such that the family $\{B_1, \dots, B_m\}$ is dominant is given by

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} (2^m - k)^n.$$

To count the number of families, we remove the order between $B_1, \dots, B_m$. Thus, the number of dominant families consisting of m sets is given by

$$\frac{1}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (2^m - k)^n.$$

Since a dominant family must have elements of $\{1, \dots, n\}$ in m distinct regions, there is no dominant family if $m > n$. Hence, we obtain the number of dominant families given by

$$\sum_{m=1}^{n} \frac{1}{m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (2^m - k)^n.$$

□

Example 3.3. Since the Dedekind numbers are known for all $n \leq 9$, we can use the formula given by Theorem 3.2 to compute $p(\mathbb{T}_M)$ explicitly, for all $n \leq 9$.

For $n = 1$, $p(\mathbb{T}_M) = \dfrac{|A_1'|}{|S_1'|} = \dfrac{1}{2}$,

For $n = 2$, $p(\mathbb{T}_M) = \dfrac{|A_2'|}{|S_2'|} = \dfrac{4}{5}$,

For $n = 3$, $p(\mathbb{T}_M) = \dfrac{|A_3'|}{|S_3'|} = \dfrac{17}{19}$,

For $n = 4$, $p(\mathbb{T}_M) = \dfrac{|A_4'|}{|S_4'|} = \dfrac{97}{167}$, For $n = 5$, $p(\mathbb{T}_M) = \dfrac{|A'_5|}{|S'_5|} = \dfrac{812}{7580}$,

For $n = 6$, $p(\mathbb{T}_M) = \dfrac{|A'_6|}{|S'_6|} = \dfrac{10127}{7828353}$,

For $n = 7$, $p(\mathbb{T}_M) = \dfrac{|A'_7|}{|S'_7|} = \dfrac{186139}{2414682040997}$,

For $n = 8$, $p(\mathbb{T}_M) = \dfrac{|A'_8|}{|S'_8|} = \dfrac{4976594}{5613043722868755 7907787}$, and

For $n = 9$, $p(\mathbb{T}_M) = \dfrac{|A'_9|}{|S'_9|} = \dfrac{191272047}{286386577668298411128469151667598498812365}$.

It is possible to obtain an expression for $|A'_n|$ in terms of Stirling numbers of the second kind (see [S, Page 81]).

Theorem 3.4.

$$|A'_n| = \sum_{k=1}^{n}\sum_{m=1}^{k}\binom{n}{k}(2^m - m)^{n-k}S(k,m).$$

Proof. By the Inclusion-Exclusion Principle, $\sum_{k=0}^{m}(-1)^k\binom{m}{k}(2^m-k)^n$ is the number of functions f from $S = \{1,\dots,n\}$ to $T = \{1,\dots,2^m\}$ whose image contains $\{1,\dots,m\}$. We obtain the same number by counting surjective functions from a k-element subset of S to $\{1,\dots,m\}$, then assigning the remaining $n-k$ numbers to $T-\{1,\dots,m\}$. Since the number of surjective functions from a k-element set to an m-element set is $S(k,m)m!$, we obtain

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}(2^m-k)^n = \sum_{k=m}^{n}\binom{n}{k}(2^m-m)^{n-k}S(k,m)m!\,.$$

Thus, we obtain

$$|A'_n| = \sum_{m=1}^{n}\sum_{k=m}^{n}\binom{n}{k}(2^m - m)^{n-k}S(k,m).$$

The result follows by interchanging the order of summations. □

We obtain upper and lower estimates of A'_n.

Theorem 3.5. *If $n \geq 2$, then*

$$\binom{n}{\lfloor n/2\rfloor}\left(2^{\lfloor n/2\rfloor} - \lfloor n/2\rfloor\right)^{\lceil n/2\rceil} \leq |A'_n| \leq 3^n 2^{\lfloor n/2\rfloor\cdot\lceil n/2\rceil}$$

Proof. The lower estimate is obtained by taking $k = m = \lfloor n/2\rfloor$.

Using $S(k,m) \leq \binom{k}{m}m^{k-m}$ (see [RD, Theorem 3]) and $m \leq 2^m - m < 2^m$, we obtain the upper estimate

$$\sum_{k=1}^{n}\sum_{m=1}^{k}\binom{n}{k}(2^m-m)^{n-k}S(k,m) \leq \sum_{k=1}^{n}\sum_{m=1}^{k}\binom{n}{k}\binom{k}{m}(2^m-m)^{n-k}m^{k-m}$$

$$\leq \sum_{k=1}^{n}\sum_{m=1}^{k}\binom{n}{k}\binom{k}{m}2^{m(n-m)} \leq \sum_{k=1}^{n}\sum_{m=1}^{k}\binom{n}{k}\binom{k}{m}2^{\lfloor n/2\rfloor\cdot\lceil n/2\rceil} \leq 3^n 2^{\lfloor n/2\rfloor\cdot\lceil n/2\rceil}.$$

□

By

$$\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil = \begin{cases} \frac{n^2}{4} & \text{if } n \text{ is even,} \\ \frac{n^2-1}{4} & \text{if } n \text{ is odd,} \end{cases}$$

we obtain an asymptotic formula for the base-2 logarithm of $|A'_n|$.

Corollary 3.6.

$$\log_2 |A'_n| = \frac{n^2}{4} + O(n).$$

Proof. By Theorem 3.5 , for sufficiently large n,

$$2^{(\lfloor n/2 \rfloor - 1)\cdot \lceil n/2 \rceil} \leq |A'_n| \leq 3^n 2^{n^2/4}.$$

Since $\lfloor n/2 \rfloor \cdot \lceil n/2 \rceil = (n/2 + O(1))(n/2 + O(1)) = n^2/4 + O(n)$ and $3^n 2^{n^2/4} = 2^{n^2/4+O(n)}$, we have the result. □

We are able to find an asymptotic formula for $|A'_n|$. We will see that the lower estimate from Theorem 3.5 is not too far from the truth.

Theorem 3.7. *For $n \geq N_0$, we have*

$$|A'_n| = \frac{C_n 2^{n^2/4+n+1/2}}{\sqrt{\pi n}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

where

$$C_n = \begin{cases} 1 + 2\sum_{k=1}^{\infty} 2^{-k^2} & \textit{if } n \textit{ is even} \\ 2\sum_{k=1}^{\infty} 2^{-(2k-1)^2/4} & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. As Theorem 3.5, we begin with

$$\sum_{k=1}^{n}\sum_{m=1}^{k}\binom{n}{k}(2^m - m)^{n-k}S(k,m) \leq \sum_{k=1}^{n}\sum_{m=1}^{k}\binom{n}{k}\binom{k}{m}(2^m - m)^{n-k}m^{k-m}$$

$$\leq \sum_{k=1}^{n}\sum_{m=1}^{k}\binom{n}{k}\binom{k}{m}2^{k(n-k)}m^{k-m}.$$

The outer sum over $|k - n/2| > \sqrt{g(n)n}$ with $g(n) = 2\log_2 n$ is negligible as follows.

$$\sum_{|k-n/2|>\sqrt{g(n)n}}\sum_{m=1}^{k}\binom{n}{k}\binom{k}{m}2^{k(n-k)}k^{k-m} \leq \sum_{k=1}^{n}\binom{n}{k}2^{n^2/4-g(n)n}(1+k)^k$$

$$\leq \sum_{k=1}^{n}\binom{n}{k}2^{n^2/4-g(n)n}(1+n)^k \leq 2^{n^2/4-g(n)n}(2+n)^n = 2^{n^2/4-2n\log_2 n + n\log_2(2+n)}.$$

The inner sum over $|m - n/2| > \sqrt{h(n)n}$ with $h(n) = 2$ is also negligible as follows.

$$\sum_{k=1}^{n}\sum_{\substack{1\leq m\leq k \\ |m-n/2|>\sqrt{h(n)n}}}\binom{n}{k}\binom{k}{m}(2^m - m)^{n-k}m^{k-m}$$

$$\leq \sum_{k=1}^{n} \sum_{\substack{1\leq m\leq k \\ |m-n/2|>\sqrt{h(n)n}}} \binom{n}{k}\binom{k}{m} 2^{m(n-m)} \leq 2^{n^2/4-h(n)n}3^n = 2^{n^2/4}4^{-n}3^n.$$

Thus, the main contribution is from

$$\sum_{|k-n/2|\leq\sqrt{g(n)n}} \sum_{\substack{1\leq m\leq k \\ |m-n/2|\leq\sqrt{h(n)n}}} \binom{n}{k}(2^m-m)^{n-k}S(k,m).$$

By $m \leq k$ in the inner sum and using $(2^m-m)/2^m = 1+O(n/2^{n/3})$ so that $(2^m-m)^{n-k} = 2^{m(n-k)}\left(1+O\left(n^2/2^{n/3}\right)\right)$, the above further boils down to

$$I_1+I_2 := \sum_{|k-n/2|\leq\sqrt{2n}} \sum_{n/2-\sqrt{2n}\leq m\leq k} \binom{n}{k} 2^{m(n-k)}S(k,m)$$

$$+ \sum_{n/2+\sqrt{2n}<k\leq n/2+\sqrt{2n\log_2 n}} \sum_{n/2-\sqrt{2n}\leq m\leq n/2+\sqrt{2n}} \binom{n}{k} 2^{m(n-k)}S(k,m)$$

The inner sum of I_2 is treated by [A, Corollary 1],

$$\sum_{n/2-\sqrt{2n}\leq m\leq n/2+\sqrt{2n}} 2^{m(n-k)}S(k,m) \leq (2^{n-k})^k \exp\left(\frac{k^2}{2\cdot 2^{n-k}}\right) \leq 2^{k(n-k)}\exp\left(\frac{n^2}{2^{n/3}}\right)$$

For large enough n so that $\exp\left(\frac{n^2}{2^{n/3}}\right) \leq 2$, the inner sum of I_2 is at most $2^{k(n-k)+1}$. Now, I_2 is at most

$$\sum_{n/2+\sqrt{2n}<k\leq n/2+\sqrt{2n\log_2 n}} \binom{n}{k} 2^{k(n-k)+1} \leq 2^{n^2/4-2n+1}\cdot 2^n = 2^{n^2/4-n+1}.$$

Then we are left with I_1. Recall

$$I_1 = \sum_{|k-n/2|\leq\sqrt{2n}} \sum_{n/2-\sqrt{2n}\leq m\leq k} \binom{n}{k} 2^{m(n-k)}S(k,m).$$

By $2^{m(n-k)} = 2^{\frac{3}{4}m(n-k)+\frac{1}{4}m(n-k)}$ and [A, Corollary 1], we obtain

$$\sum_{m\leq n/2-\sqrt{2n}} 2^{m(n-k)}S(k,m) \leq 2^{\frac{3}{4}(n^2/4-2n)} \sum_{m\leq n/2-\sqrt{2n}} 2^{\frac{1}{4}m(n-k)}S(k,m)$$

$$\leq 2^{\frac{3}{4}(n^2/4-2n)}(2^{\frac{n-k}{4}})^k \exp\left(\frac{k^2}{2\cdot 2^{(n-k)/4}}\right) \leq 2^{\frac{3}{4}(n^2/4-2n)}2^{\frac{k(n-k)}{4}}\exp\left(\frac{n^2}{2^{n/12}}\right)$$

$$\leq 2^{\frac{3}{4}(n^2/4-2n)}2^{\frac{1}{4}(n^2/4)+1} = 2^{n^2/4-3n/2+1}.$$

This yields

$$\sum_{|k-n/2|\leq\sqrt{2n}} \sum_{m\leq n/2-\sqrt{2n}} \binom{n}{k} 2^{m(n-k)}S(k,m) \leq 2^{n^2/4-3n/2+1}2^n = 2^{n^2/4-n/2+1}.$$

Then

$$I_1 = \sum_{|k-n/2|\leq\sqrt{2n}} \sum_{1\leq m\leq k} \binom{n}{k} 2^{m(n-k)}S(k,m) + O(2^{n^2/4-n/2+1}).$$

By [A, Corollary 1] and [A, (8)], we have

$$2^{k(n-k)}\left(1+\frac{k(k-1)}{2\cdot 2^{n-k}}\right)\leq \sum_{1\leq m\leq k} 2^{m(n-k)}S(k,m)\leq 2^{k(n-k)}\exp\left(\frac{k^2}{2\cdot 2^{n-k}}\right).$$

Thus,

$$\sum_{1\leq m\leq k} 2^{m(n-k)}S(k,m)=2^{k(n-k)}\left(1+O\left(\frac{n^2}{2^{n/3}}\right)\right).$$

The main contribution to I_1 is now

$$\sum_{|k-n/2|\leq\sqrt{2n}}\binom{n}{k}2^{k(n-k)}.$$

We apply the following [KS, Lemma 5.2] which gives an estimate of binomial coefficients by Stirling's formula:

$$\frac{1}{2^n}\binom{n}{n/2+r(n)\sqrt{n}}=\frac{2}{\sqrt{2\pi n}}\exp(-2(r(n))^2)\left(1+O\left(\frac{(\log n)^3}{\sqrt{n}}\right)\right)$$

provided that $|r(n)|\leq 6\log n$.

Further splitting the sum over $k=n/2+r(n)\sqrt{n}$ into two ranges according to $|r(n)|\leq 1/\sqrt[4]{n}$ or $|r(n)|> 1/\sqrt[4]{n}$, the latter sum is negligible

$$\sum_{\sqrt[4]{n}<|k-n/2|\leq\sqrt{2n}}\binom{n}{k}2^{k(n-k)}=O\left(\frac{2^n}{\sqrt{n}}\cdot 2^{n^2/4-\sqrt{n}}\right).$$

The former sum is the main contribution

$$\sum_{|k-n/2|\leq\sqrt[4]{n}}\binom{n}{k}2^{k(n-k)}=\frac{2^{n+1}}{\sqrt{2\pi n}}\left(1+O\left(\frac{(\log n)^3}{\sqrt{n}}\right)\right)2^{n^2/4}(C_n+O(2^{-\sqrt{n}})).$$

In this range $|r(n)|\leq 1/\sqrt[4]{n}$, we may improve the term $(\log n)^3/\sqrt{n}$ by $1/\sqrt{n}$ in [KS, Lemma 5.2]. The result follows after simplifying the estimate. □

Theorem 3.5 and Proposition 3.1 are enough to conclude that $p(\mathbb{T}_M)$ approaches zero as $n\to\infty$.

Corollary 3.8. *We have for sufficiently large n,*

$$p(\mathbb{T}_M)\leq\frac{3^n2^{n^2/4}}{2^{\binom{n}{\lfloor n/2\rfloor}}-1}\leq\frac{2^{n^2}}{2^{2^n/n}}$$

which approaches zero as $n\to\infty$.

4. Final Comments

We close this article by suggesting how the present work can be extended naturally to other classes of monomial ideals. It would be interesting to compute the probability that a monomial ideal is close to being dominant. This idea of proximity can be interpreted in two different ways as we explain next.

A monomial ideal can be close to being dominant in how we construct it. In [Al, Al1], we introduced the class of semidominant ideals. These ideals have the property that all of its minimal generators but one satisfy the definition of dominance. Thus, semidominant ideals are almost dominant in how they are constructed.

But a monomial ideal can be close to being dominant in terms of its combinatorial properties. Recall that the minimal resolution of a monomial ideal can be obtained from its Taylor resolution by means of consecutive cancellations. Since the minimal resolution of a dominant ideal can be obtained from its Taylor resolution by doing zero cancellations (for the Taylor resolution of a dominant ideal is already minimal), we can say that a monomial ideal is close to being dominant if its minimal resolution can be obtained from its Taylor resolution after doing exactly one consecutive cancellation. This class is called the class of 1-cancellation ideals [Al1].

A nice characterization of the 1-cancellation ideals can be found in [Al1, Theorem 3.1]. We believe that a natural way to extend this paper is by studying the probability that a randomly chosen monomial ideal is either a 1-cancellation ideal or a semidominant ideal.

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