# SOLUTIONS, MATH 214B

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**Problem1**.
(a) Define $i : \mathcal{F} \longrightarrow \widetilde{\mathcal{F}}$ by

$$
\begin{aligned}
i_U : & \mathcal{F}(U) \longrightarrow \widetilde{\mathcal{F}}(U) \\
& a \mapsto \begin{pmatrix} i_U(a) : U \to Tot\widetilde{\mathcal{F}} \\ x \mapsto a_x \end{pmatrix}.
\end{aligned}
$$

Remark that $i_U(a) : U \to Tot\widetilde{\mathcal{F}}$ is a continuous section. Let $U_1 \subset U_2 \subset X$ be open subsets of $X$. Consider the below diagram:

$$
\begin{array}{ccc}
\mathcal{F}(U_2) & \xrightarrow{\quad i_{U_2} \quad} & \widetilde{\mathcal{F}}(U_2) \\
\Big\downarrow {\scriptstyle \mathcal{F}(1_{U_2U_1})} & = & \Big\downarrow {\scriptstyle \widetilde{\mathcal{F}}(1_{U_2U_1})} \\
\mathcal{F}(U_1) & \xrightarrow{\quad i_{U_1} \quad} & \widetilde{\mathcal{F}}(U_1)
\end{array}
$$

Then for $a \in \mathcal{F}(U_2)$, we have $\widetilde{\mathcal{F}}(1_{U_2U_1}) \circ i_{U_2}(a)(x) = a_x$ for $x \in U_1$. On the other hand, $i_{U_1} \circ \mathcal{F}(1_{U_2U_1})(a)(x) = (a|_{U_1})_x$ for $x \in U_1$. Indeed, the definition of stalks gives $a_x = (a|_{U_1})_x$. Hence, the diagram commutes and $i$ is a natural morphism of presheaves $i : \mathcal{F} \to \widetilde{\mathcal{F}}$.

(b) $\Leftarrow$) This part is obvious, since $\widetilde{\mathcal{F}}$ is a sheaf.

$\Rightarrow$) Suppose that $\mathcal{F}$ is a sheaf. We want to find $j : \widetilde{\mathcal{F}} \to \mathcal{F}$ with $j \circ i = 1_{\mathcal{F}}$, and $i \circ j = 1_{\widetilde{\mathcal{F}}}$. For any continuous section $s$ of $p$ and any open set $U \subset X$, we have an open cover of $U$,

$$
U = \bigcup_{a \in \mathcal{F}(U)} U_a \tag{1}
$$

where $U_a = s^{-1}(\{a_x | x \in U\})$. We claim that there exist $j(s) \in \mathcal{F}(U)$ such that $j(s)|_{U_a} = a$ for each $a \in \mathcal{F}(U)$. In fact, $x \in U_a \bigcap U_b$ implies $a_x = b_x$. From SHEAF(2), we obtain $a|_{U_a \bigcap U_b} = b|_{U_a \bigcap U_b}$, and hence we obtain the existence of $j(s)$ such that $j(s)|_{U_a} = a$ by SHEAF(3). Furthermore, this $j(s)$ is uniquely determined by SHEAF(2).

Now, we show that $j \circ i = 1_{\mathcal{F}}$. Let $U \subset X$ be open, and $a \in \mathcal{F}(U)$. From $i(a)^{-1}(\{a_x | x \in U\}) = U$, we obtain $ji(a) = a$. It remains to show that $i \circ j = 1_{\widetilde{\mathcal{F}}}$. We use the open cover of $U = \bigcup U_a$ again. For $s \in \widetilde{\mathcal{F}}(U)$, $s(x) = a_x$ for $x \in U_a$. Definition of $i$ in (a) implies $ij(s)(x) = j(s)_x = a_x = s(x)$. Hence, now SHEAF(2) implies that $ij(s) = s$.

(c) (Existence) We use (b) on $\mathcal{G}$ to obtain an isomorphism $i_G : \mathcal{G} \to \widetilde{\mathcal{G}}$, then it suffices to find $\widetilde{j}' : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ such that $i_G^{-1} \circ \widetilde{j}' \circ i = j$. Define $\widetilde{j}' : \widetilde{\mathcal{F}} \longrightarrow \widetilde{\mathcal{G}}$ by

$$\widetilde{j}'_U : \widetilde{\mathcal{F}}(U) \longrightarrow \widetilde{\mathcal{G}}(U)$$

$$\begin{pmatrix} s : U \to Tot\widetilde{\mathcal{F}} \\ x \mapsto [U, a] \in \mathcal{F}_x \end{pmatrix} \mapsto \begin{pmatrix} \widetilde{j}'_U(s) : U \to Tot\widetilde{\mathcal{G}} \\ x \mapsto [U, j(a)] \in \mathcal{G}_x \end{pmatrix}.$$

Then, the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{F}}(U_2) & \xrightarrow{\widetilde{j}'} & \widetilde{\mathcal{G}}(U_2) \\ {\scriptstyle res}\downarrow & = & \downarrow{\scriptstyle res} \\ \widetilde{\mathcal{F}}(U_1) & \xrightarrow{\widetilde{j}'} & \widetilde{\mathcal{G}}(U_1) \end{array}$$

commutes where $U_1 \subset U_2 \subset X$. Let $\widetilde{j} = i_G^{-1} \circ \widetilde{j}'$.
(Uniqueness) Let $U \subset X$ be open. We use the open cover in (1) in (b) again, $U = \bigcup U_a$. For any $s \in \widetilde{\mathcal{F}}(U)$, we have $i(a|_{U_a}) = s|_{U_a}$. The condition $\widetilde{j} \circ i = j$ forces $\widetilde{j}(s|_{U_a}) = \widetilde{j}(s)|_{U_a} = j(a|_{U_a})$. Hence by SHEAF(2), $\widetilde{j}(s)$ is uniquely determined.

**Problem2**.
(a) SHEAF(1): $\mathcal{G}(\phi) = \{\phi\}$ is a final object in the category SETS.
SHEAF(2): Let $U = \bigcup U_\alpha$ be an open cover. Let $a, b \in \mathcal{G}(U)$ with $a|_{U_\alpha} = b|_{U_\alpha}$ for all $\alpha$. For any $x \in U$, there is some $\alpha$ such that $x \in U_\alpha$. Since $a|_{U_\alpha} = b|_{U_\alpha}$, we have

$$a(x) = a|_{U_\alpha}(x) = b|_{U_\alpha}(x) = b(x).$$

Hence, $a = b$.
SHEAF(3): Let $U = \bigcup U_\alpha$ be an open cover. Let $a_\alpha \in \mathcal{G}(U_\alpha)$ satisfy

$$a_\alpha|_{U_\alpha \bigcap U_\beta} = a_\beta|_{U_\alpha \bigcap U_\beta}$$

for any $\alpha, \beta$. We define $a \in \mathcal{G}(U)$ by

$$a(x) = a_\alpha(x)$$

where $x \in U_\alpha$. Then we have $a|_{U_\alpha} = a_\alpha$.

(b) The natural morphism $i : \mathcal{F} \to \mathcal{G}$ is defined by

$$\begin{aligned} i_U : &\mathcal{F}(U) = A \longrightarrow \mathcal{G}(U) \\ &a \in A \mapsto \begin{pmatrix} i_U(a) : U \to A \\ x \mapsto a \end{pmatrix}. \end{aligned}$$

Recall from Problem1 (a) that $i : \mathcal{F} \longrightarrow \widetilde{\mathcal{F}}$ is defined by

$$\begin{aligned} i_U : &\mathcal{F}(U) \longrightarrow \widetilde{\mathcal{F}}(U) \\ &a \mapsto \begin{pmatrix} i_U(a) : U \to Tot\widetilde{\mathcal{F}} \\ x \mapsto a_x \end{pmatrix}. \end{aligned}$$

We show that these morphisms are isomorphic by showing that

$$f : \mathcal{G}(U) \longrightarrow \widetilde{\mathcal{F}}(U)$$
$$(s : U \to A) \mapsto \begin{pmatrix} \tilde{s} : U \to Tot\mathcal{F} \\ x \mapsto s(x)_x \end{pmatrix}$$

is an isomorphism of sets.
($f$ is injective) Suppose $s_1(x) \neq s_2(x)$ for some $x \in U$, then $s_1(x)|_x \neq s_2(x)|_x$. Thus, $f(s_1) \neq f(s_2)$.
($f$ is surjective) Let $\tilde{s} : U \to Tot\mathcal{F}$ be a continuous section. For each $x \in U$, define $s(x) \in A$ by $\tilde{s}(x) = s(x)|_x$. Then, for fixed $a \in A$, we have

$$\{x \in U | s(x) = a\} = \{x \in U | \tilde{s}(x) \in \{a_x | x \in U\}\}.$$

The RHS is an open set since $\tilde{s}$ is continuous, thus LHS is also an open set in $U$. Since this is true for all $a \in A$, we conclude that $s$ is continuous and $f(s) = \tilde{s}$. Hence the natural morphism $i : \mathcal{F} \to \mathcal{G}$ is isomorphic to $i : \mathcal{F} \to \widetilde{\mathcal{F}}$.

**Problem3**.
Let $P$ be a nonzero prime ideal in $A = \mathbb{Z}[X]$. Then the natural homomorphism $\mathbb{Z} \longrightarrow A/P$ has kernel $P \cap \mathbb{Z}$. This gives an embedding of $\mathbb{Z}/(P \cap \mathbb{Z})$ into $A/P$. Since $A/P$ is an integral domain, so is $\mathbb{Z}/(P \cap \mathbb{Z})$. Thus, we have two cases

$$P \cap \mathbb{Z} = \begin{cases} p\mathbb{Z} & \text{for some prime } p \in \mathbb{Z} \\ (0) \end{cases}$$

(Case1) $P \cap \mathbb{Z} = p\mathbb{Z}$ for some prime $p \in \mathbb{Z}$:
By 3rd isomorphism theorem, we have

$$A/P \simeq (\mathbb{Z}/p\mathbb{Z}[X]) / (P/p\mathbb{Z}[X]).$$

In fact $\mathbb{Z}/p\mathbb{Z}[X] = \mathbb{F}_p[X]$, and the LHS is an integral domain. It follows that $P/p\mathbb{Z}[X]$ is a prime ideal in $\mathbb{F}_p[X]$. Since $\mathbb{F}_p[X]$ is UFD, $P/p\mathbb{Z}[X] = (f(X))$ for some $f(X) \in \mathbb{F}_p[X]$ irreducible polynomial of degree $\geq 1$ or $P/p\mathbb{Z}[X] = (0)$. Hence, in this case, we obtain $P = (p, f(X))$ or $P = p\mathbb{Z}[X]$ where $f$ is irreducible mod $p$.
(Case2) $P \cap \mathbb{Z} = (0)$:
Consider the ideal $P\mathbb{Q}[X] \subset \mathbb{Q}[X]$, this is a proper prime ideal in $\mathbb{Q}[X]$. So, $P\mathbb{Q}[X] = f(X)\mathbb{Q}[X]$ where $f$ is irreducible over $\mathbb{Q}$. Further, we can assume that the polynomial $f$ is primitive. We claim that $P = f(X)\mathbb{Z}[X]$. Suppose $h \in P$, $h = fg$ for some $g \in \mathbb{Q}[X]$. Taking content(Gauss lemma) on each side, we obtain $g \in \mathbb{Z}[X]$. Hence it follows that $P = f(X)\mathbb{Z}[X]$.
Now, we can write the result as follows:
Prime ideals $P$ in $\mathbb{Z}[X]$ are one of the following forms:

$$(2) \qquad P = \begin{cases} (0), \\ (f(X)) & \text{for } f \in \mathbb{Z}[X] \text{ irreducible and primitive,} \\ (p) & \text{for some prime } p \in \mathbb{Z}, \\ (p, f(X)) & \text{for some prime } p \in \mathbb{Z}, \text{ and } f \text{ is irreducible mod } p. \end{cases}$$

Now, we characterize the topology on Spec($\mathbb{Z}[X]$). Let $I \subset \mathbb{Z}[X]$ be a proper ideal. Consider $I \cap \mathbb{Z} = n\mathbb{Z}$, we have two cases,
(Case1) $I \cap \mathbb{Z} = n\mathbb{Z}$ with $n \neq 0, \pm 1$:

Let $\mathfrak{p} \in V(I)$, i.e. $\mathfrak{p}$ is a prime ideal containing $I$. Then $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$ for some prime $p|n$. Fix a prime $p|n$. The ideal $I + p\mathbb{Z}[X] \subset \mathbb{Z}[X]$ maps to some ideal $(f(X)) \subset \mathbb{F}_p[X]$ by reducing mod $p$, since $\mathbb{F}_p[X]$ is a PID. Let $f_i$ be distinct irreducible factors of $f$ in $\mathbb{F}_p[X]$ if $\deg(f) > 0$ , and enumeration of all irreducible polynomials of $\mathbb{F}_p[X]$ with 0 if $f = 0$. Thus, we have $(f(X)) \subset (f_i(X)) \subset \mathbb{F}_p[X]$ for each $i$. Pulling back these ideals to $\mathbb{Z}[X]$, we obtain $I \subset I + p\mathbb{Z}[X] \subset (p, f_i(X)) \subset \mathbb{Z}[X]$ for each $i$. Hence, the result

$$\begin{aligned} V(I) = \{(p, f_{p,i}) \mid\ & p|n, (I + p\mathbb{Z}[X])/(p\mathbb{Z}[X]) = (f_p(X)) \subset \mathbb{F}_p[X], \deg(f_p) > 0, f_{p,i} \text{ are} \\ & \text{distinct irreducible factor of } f \text{ in } \mathbb{F}_p[X]\} \\ & \bigcup \{(p, f_{p,i}) \mid p|n, (I + p\mathbb{Z}[X])/(p\mathbb{Z}[X]) = (0) \subset \mathbb{F}_p[X], f_{p,i} \text{ are enumeration of} \\ & \text{all irreducible polynomials of } \mathbb{F}_p[X] \text{ with } 0\}. \end{aligned}$$

(Case2) $I \cap \mathbb{Z} = (0)$:
Consider $I\mathbb{Q}[X] = (f(X)) \subsetneq \mathbb{Q}[X]$ with $f$ being primitive. Then we obtain $I = f(X)\mathbb{Z}[X]$ by Gauss lemma. Let $f_i$ be distinct irreducible factors of $f$, and $\tau_i \in \mathbb{C}$ be the corresponding roots of $f_i$. For each $i$, we need to find primes $p$ such that $(p, f_i)$ become proper. To do this, we use Gauss lemma again so that we obtain the result:

$$(p, f_i) \text{ is proper } \iff \frac{1}{p} \notin \mathbb{Z}[\tau_i].$$

Hence, we have,

$$\begin{aligned} V(I) = \{(p, f_{i,j}) \mid f_i|f, \frac{1}{p} \notin \mathbb{Z}[\tau_i], ((f_i) + p\mathbb{Z}[X])/(p\mathbb{Z}[X]) = (f_i(X) \bmod p) \subset \mathbb{F}_p[X], \\ \deg(f_i \bmod p) > 0, f_{i,j} \text{ are distinct irreducible factor of } f_i \text{ in } \mathbb{F}_p[X]\} \\ \bigcup \{(f_i) |\ f_i|f \text{ irreducible}\}. \end{aligned}$$

**Problem4**.
Let $X = Y = \mathrm{Spec}\mathbb{C}$, $S = \mathrm{Spec}\mathbb{R}$. They are all affine schemes. Also, $\mathbb{C}$ can be regarded as $\mathbb{R}$-algebra. Then $Z = \mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ is the desired pull-back of the diagram,

$$\begin{array}{ccc} Z & \longrightarrow & \mathrm{Spec}\mathbb{C} \\ \downarrow & = & \downarrow \\ \mathrm{Spec}\mathbb{C} & \longrightarrow & \mathrm{Spec}\mathbb{R} \end{array}$$

Since $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\mathbb{C} \times \mathbb{C}$ as rings, we have $Z = \mathrm{Spec}(\mathbb{C} \times \mathbb{C})$. Hence, we obtain the result

$$Z = \mathrm{Spec}\mathbb{C} \times_{\mathrm{Spec}\mathbb{R}} \mathrm{Spec}\mathbb{C} = \{\mathbb{C} \times \{0\}, \{0\} \times \mathbb{C}\}.$$

**Problem6**.
$\Leftarrow$) Suppose first that $A$ has a nontrivial idempotent $a$. Then, we claim that $A = aA \oplus (1-a)A$. For any $x \in A$, $x = ax + (1-a)x$, so $A = aA + (1-a)A$. If $y \in aA \cap (1-a)A$, then $ay \in a(1-a)A = 0$, and $(1-a)y \in (1-a)aA = 0$. Thus, $y = 0$ and $A = aA \oplus (1-a)A$.

Now, we have $V(aA) \cap V((1-a)A) = V(aA + (1-a)A) = V(A) = \emptyset$, and $V(aA) \cup V((1-a)A) = V(aA \cap (1-a)A) = V((0)) = \text{Spec}(A)$. Hence, Spec$(A)$ is covered by disjoint union of nonempty closed sets $V(aA)$ and $V((1-a)A)$, i.e. Spec$(A)$ is disconnected.

$\Rightarrow$) Suppose that Spec$(A)$ is disconnected, i.e. $\text{Spec}(A) = V(J_1) \cup V(J_2)$, $V(J_1) \cap V(J_2) = \emptyset$ for some ideals $J_1, J_2 \subset A$, and $V(J_1) \neq \emptyset$, $V(J_2) \neq \emptyset$. If $I, J$ are radical ideals, then we have $V(I) = V(J) \iff I = J$. Take $I_1 = \sqrt{J_1}$ and $I_2 = \sqrt{J_2}$. Then, we obtain $I_1 + I_2 = A$ and $I_1 \cap I_2 = \sqrt{(0)}$. Also, we know that $I_1$, $I_2$ are proper. Thus, we can find $a \in I_1$, $b \in I_2$ such that $a + b = 1$. However $ab \in I_1 I_2 = \sqrt{(0)}$, so we see that $(ab)^n = 0$ for some $n \geq 1$. Using $(a+b)^{2n} = 1$, we obtain $a^n A + b^n A = A$. Let $a' \in a^n A$, and $b' \in b^n A$ with $a' + b' = 1$. Then, $a'$ is the desired nontrivial idempotent, since $a' = a'(a' + b') = a'^2 + a'b' = a'^2$ implies $a'^2 = a'$.

**Problem7**.
Remark that $f(P)$ is the image of $f$ in the residue field $A_P/(PA_P)$ where $A_P$ is the localization. We claim that $f(P) = 0$ if and only if $f \in P$. We see that

$$\begin{aligned} f(P) = 0 &\iff \text{There exists } s \in S = A - P \text{ such that } fs/s \in PA_P \\ &\iff \text{There exists } s', s'' \in S \text{ and } p \in P \text{ such that } (p - fs')ss'' = 0 \\ &\iff f \in P \end{aligned}$$

, since $s'ss'' \notin P$. Hence, the set $\{P \in \text{Spec}(A) |\ f(P) = 0\}$ is just $V(fA)$, so it is closed.

**Problem8**.
Take $A = \mathbb{Z}[X]$, and $U = D(2) \cup D(x) = \text{Spec}(A) - (V(2A) \cap V(xA))$. Suppose $U = D(a)$ for some $a \in A$. Then,

$$\begin{aligned} U = D(a) &\iff \text{Spec}(A) - V(aA) = \text{Spec}(A) - V((2, x)) \\ &\iff V(aA) = V((2, x)) = \{(2, x)\}. \end{aligned}$$

, since $(2, x)$ is maximal ideal. Thus, for a prime ideal $\mathfrak{p} \in \text{Spec}(A)$, we have $\mathfrak{p} \supset aA \iff \mathfrak{p} = (2, x)$. Suppose $\deg(a) > 0$, then we can find an irreducible factor $b \in A$ of $a$. Further, as in problem3, we can find a prime number $p \in \mathbb{Z}$ such that $(p, b)$ is proper. Then, we obtain $V(aA) \supset \{(b), \mathfrak{m}\}$, where $\mathfrak{m}$ is a maximal ideal that contains $(p, b)$, and this is a contradiction to $V(aA) = \{(2, x)\}$. Now, we assume that $a \in \mathbb{Z}$. Our assumption implies that $a$ cannot be unit or zero. Then, there is a prime number $p \in \mathbb{Z}$ such that $V(aA) \supset \{(p), (p, x)\}$. This again contradicts $V(aA) = \{(2, x)\}$. Hence $U = D(a)$ is impossible for any $a \in A$.

**Problem9**.
First, consider $Mor_{Rings}(\mathbb{Z}, \mathbb{Q}) = \{i : \mathbb{Z} \hookrightarrow \mathbb{Q}\}$, and $Mor_{TopSpaces}(\text{Spec}\mathbb{Q}, \text{Spec}\mathbb{Z}) = \{f : \text{Spec}\mathbb{Q} \longrightarrow \text{Spec}\mathbb{Z} |\ f \text{ is continuous}\}$. We have only one point in $Mor_{Rings}(\mathbb{Z}, \mathbb{Q})$, but $Mor_{TopSpaces}(\text{Spec}\mathbb{Q}, \text{Spec}\mathbb{Z})$ contains infinitely many points, since it contains $f_p : (0) \subset \mathbb{Q} \mapsto p\mathbb{Z} \subset \mathbb{Z}$ for all prime $p \in \mathbb{Z}$. Thus, the functor Spec$(-)$ is not full.

Then we consider $Mor_{Rings}(\mathbb{C}, \mathbb{C}) \supset \{i, c\}$, where $i$ is identity, $c$ is complex conjugation. Also, consider $Mor_{TopSpaces}(\text{Spec}\mathbb{C}, \text{Spec}\mathbb{C}) = \{i_0\}$, where $i_0 : (0) \subset \mathbb{C} \mapsto (0) \subset \mathbb{C}$. We have at least two points in $Mor_{Rings}(\mathbb{C}, \mathbb{C})$, but we have only one point in $Mor_{TopSpaces}(\mathrm{Spec}\mathbb{C}, \mathrm{Spec}\mathbb{C})$. Hence, the functor $\mathrm{Spec}(-)$ is not faithful.

**Problem12**.
SHEAF(1): $\mathcal{F}(\phi) = \{\phi\}$ is a final object in the category SETS.
SHEAF(2): Let $U = \bigcup U_\alpha$ be an open cover. Let $a, b \in \mathcal{F}(U)$ with $a|_{U_\alpha} = b|_{U_\alpha}$ for all $\alpha$. For any $x \in U$, there is some $\alpha$ such that $x \in U_\alpha$. Since $a|_{U_\alpha} = b|_{U_\alpha}$, we have

$$a(x) = a|_{U_\alpha}(x) = b|_{U_\alpha}(x) = b(x).$$

Hence, $a = b$.
SHEAF(3): Let $U = \bigcup U_\alpha$ be an open cover. Let $a_\alpha \in \mathcal{F}(U_\alpha)$ satisfy

$$a_\alpha|_{U_\alpha \bigcap U_\beta} = a_\beta|_{U_\alpha \bigcap U_\beta}$$

for any $\alpha, \beta$. We define $a \in \mathcal{F}(U)$ by

$$a(x) = a_\alpha(x)$$

where $x \in U_\alpha$. Then we have $a|_{U_\alpha} = a_\alpha$.
Thus, $\mathcal{F}$ is a sheaf.
We claim that $(X, \mathcal{F})$ is a scheme. First, consider $\mathcal{F}_x = \{[U, a] | x \in U \subset_{\text{open}} X,\ a \in \mathcal{F}(U)\}$. Since $X$ is a discrete topological space, we can further show that $\mathcal{F}_x = \{[\{x\}, a] | a \in \mathcal{F}(\{x\})\} \simeq k$. This shows that $(X, \mathcal{F})$ is a local ringed space. For any $x \in X$, $U = \{x\}$, we have $(\{x\}, \mathcal{F}|_{\{x\}}) \simeq (\mathrm{Spec}k, \mathcal{O}_k)$. This proves our claim.
Suppose that $(X, \mathcal{F}) \simeq (\mathrm{Spec}A, \mathcal{O}_A)$. For any $\mathfrak{p} \in \mathrm{Spec}A$, $(\mathcal{O}_A)_\mathfrak{p} = A_\mathfrak{p} = k$. Since $A_\mathfrak{p}$ is a local ring with a unique maximal ideal $\mathfrak{p}A_\mathfrak{p}$ and $k$ is a field, we must have $\mathfrak{p} = 0$. Thus, $\mathrm{Spec}A = \{0\}$, and $A$ cannot have nonunit element, otherwise $\mathrm{Spec}A$ would contain nonzero maximal ideal of $A$. It follows that $A$ is a field, and $(X, \mathcal{F})$ is affine if and only if $X$ is a singleton set.

**Problem13**.
(a) We remark that for any $f \in K$, there is $n \in \mathbb{N}$ such that $f \in k(X_1, \cdots, X_n)$. So, there is $n \in \mathbb{N}$ such that $a(f) = (a_1, a_2, \cdots)$ with $a_i = 0$ for $i \geq n$. Also, for any $f, g \in K - \{0\}$, we have $a(fg) = a(f) + a(g)$, $a(f+g) \geq \min(a(f), a(g))$, where the addition is componentwise. Then, it follows that $\{f \in K | a(f) = 0\}$ is the set of units in $A$ and $\{f \in K | a(f) > 0\}$ forms the ideal of all nonunits in $A$. Further, we obtain that if $a(f) = (a_1, a_2, \cdots, a_n, 0, 0, \cdots)$, then $f = uX_1^{a_1}X_2^{a_2}\cdots X_n^{a_n}$ for some unit $u \in A$. Now, we claim that $Q := \sum_{i \in \mathbb{N}} X_i A = \{f \in K | a(f) > 0\}$. The inclusion $\supset$ is clear. To prove $\subset$, let $f = X_1 f_2 + \cdots + X_m f_m$. Then, $a(f) \geq \min(a(X_1 f_1), \cdots, a(X_m f_m)) > 0$. Hence, the claim is proved and $Q$ is the unique maximal ideal of $A$.

(b) The inclusion $P_i \subset P_{i+1}$ is clear for all $i \geq 1$. Also, $P_i \subset Q$ is obvious, since $Q = \{f \in A | a(f) > 0\}$. To show that $P_i$ is a prime ideal in $A$, let $f, g \in A - P_i$. Then, for some $b(f)_{i+1}, b(f)_{i+2}, \cdots$, and $b(g)_{i+1}, b(g)_{i+2}, \cdots$, we have $a(f) \leq (0, \cdots, 0, b(f)_{i+1}, b(f)_{i+2}, \cdots)$, and $a(g) \leq (0, \cdots, 0, b(g)_{i+1}, b(g)_{i+2}, \cdots)$. Adding these up, we obtain

$$a(fg) = a(f) + a(g) \leq (0, \cdots, 0, b(f)_{i+1} + b(g)_{i+1}, b(f)_{i+2} + b(g)_{i+2}, \cdots).$$

This implies $fg \in A - P_i$. Hence, $P_i$ is a prime ideal in $A$. Furthermore, the same argument as in (a) shows that $P_i = \sum_{j \leq i} X_j A$.

(c) Let $P$ be a prime ideal in $A$. Let $f \in P - \{0\}$, and $f = uX_1^{a_1} \cdots X_n^{a_n}$ for some unit $u \in A$. Since $P$ is a prime ideal, we can find $i \leq n$ such that $X_i \in P$. Define a set $B = \{i \in \mathbb{N} | X_i \in P\}$. We divide into two cases:
(Case1) $B$ is infinite:
We can show that $P$ contains all $X_n$ from the formula(*),

$$X_i = X_j \frac{X_i + X_j}{X_j} - X_j$$

for $i < j$. Hence, we obtain $P = Q$.
(Case2) $B$ is finite:
By the formula(*), there exists $n \in N$ such that $B = \{i \in \mathbb{N} | 1 \leq i \leq n\}$. Hence, it follows that $P = \sum_{i \leq n} X_i A = P_n$.
We proved that $\mathrm{Spec}A = \{0, Q, P_1, P_2, \cdots, P_n, \cdots\}$.
The Zariski topology $\mathcal{T}$ on $\mathrm{Spec}A$ is $\mathcal{T} = \{\emptyset\} \cup \{V(\mathfrak{p}) | \mathfrak{p} \in \mathrm{Spec}A\}$. To prove this, let $I$ be a proper ideal in $A$. Then, consider $m = \min\{n \in \mathbb{N} | I \subset P_n\}$. If $m \in \mathbb{N}$, then, $V(I) = V(P_m)$. If $m = \infty$, then $V(I) = V(Q) = \{Q\}$. In fact, $V(P_n) = \{Q, P_n, P_{n+1}, \cdots\}$ for each $n \geq 1$.

(d) The topology on $\mathrm{Spec}A - \{Q\}$ is the subspace topology

$$\mathcal{T}' = \{\emptyset\} \cup \{V(\mathfrak{p}) - \{Q\} | \mathfrak{p} \in \mathrm{Spec}A\}.$$

For each point $P_n \in \mathrm{Spec}A - \{Q\}$, we have $\overline{\{P_n\}} = V(P_n) - \{Q\} \neq \{P_n\}$. For $0 \in \mathrm{Spec}A - \{Q\}$, $\overline{\{0\}} = V(0) - \{Q\} \neq \{0\}$. Hence, the scheme $\mathrm{Spec}A - \{Q\}$ has no closed points.

**Problem14**.
As a map of topological spaces, it is clear that $X \xrightarrow{f} Y$ factors through $X \xrightarrow{f} U \xrightarrow{i} Y$, where $U \xrightarrow{i} Y$ is the inclusion. Let $f(x) = y$, we have composition of morphisms of schemes $(X, \mathcal{F}) \xrightarrow{f} (U, \mathcal{G}|_U) \xrightarrow{i} (Y, \mathcal{G})$. This induces morphisms of local rings $\mathcal{G}_y \xrightarrow{id} (\mathcal{G}|_U)_y \longrightarrow \mathcal{F}_x$, since $y \in U$. Further, we know that the ring homomorphism $\mathcal{G}_y \longrightarrow \mathcal{F}_x$ is a local. Thus, $\mathcal{G}_y \xrightarrow{id} (\mathcal{G}|_U)_y \longrightarrow \mathcal{F}_x$ is a composition of local ring homomorphisms. Hence the morphism of schemes $f$ factors through $X \longrightarrow U$ and $U \hookrightarrow Y$.

**Problem16**.
Let $(X, \mathcal{O}_X) \xrightarrow{f} (\mathrm{Spec}\mathbb{Z}, \mathcal{O}_\mathbb{Z})$ be a morphism of schemes, and let $f(x) = y$. Then, we have a local ring homomorphism $\mathcal{O}_{\mathbb{Z},y} \longrightarrow \mathcal{O}_{X,x}$. We have two cases,
(Case1) $y = 0$:
Since $\mathcal{O}_{\mathbb{Z},0} = \mathbb{Z}_{(0)} \longrightarrow \mathcal{O}_{X,x}$ is local, $\mathcal{O}_{\mathbb{Z},0} = \mathbb{Z}_{(0)} = \mathbb{Q} \hookrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k(x)$ where $\mathfrak{m}_{X,x}$ is the unique maximal ideal of $\mathcal{O}_{X,x}$. Thus, characteristic of $k(x)$ is 0.
(Case2) $y = p\mathbb{Z}$:
$\mathcal{O}_{\mathbb{Z},p\mathbb{Z}} = \mathbb{Z}_{(p)} \longrightarrow \mathcal{O}_{X,x}$. Since the homomorphism is local, we have $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \hookrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k(x)$. Since $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} = \mathbb{Z}/p\mathbb{Z}$, we have $\mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k(x)$. Thus, characteristic of $k(x)$ is $p$.
Hence, in either case, we have $f(x) = p\mathbb{Z}$, where $p$ is the characteristic of the residue field $k(x)$.

**Problem19**.

We remark that the scheme structure on Proj$S$ is given as follows. For each $\mathfrak{p} \in \text{Proj}S$, we consider the ring $S_{(\mathfrak{p})}$ of degree zero in the localized ring $T^{-1}S$, where $T$ is the multiplicative system consisting of all homogeneous elements of $S$ which are not in $\mathfrak{p}$. For any open subset $U \subset \text{Proj}S$, we define $\mathcal{O}(U)$ to be the set of functions $s : U \longrightarrow \coprod S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, and such that: for each $\mathfrak{p} \in U$, there exists a neighborhood $V$ of $\mathfrak{p}$ in $U$, and homogeneous elements $a, f$ in $S$, of the same degree, such that for all $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $S_{(\mathfrak{q})}$. This $\mathcal{O}$ is a sheaf and $(\text{Proj}S, \mathcal{O})$ is a scheme. Furthermore, for any $\mathfrak{p} \in \text{Proj}S$, the stalk $\mathcal{O}_{\mathfrak{p}}$ is isomorphic to the local ring $S_{(\mathfrak{p})}$.

We define $f : (\text{Proj}S, \mathcal{O}) \longrightarrow (\text{Spec}S_0, \mathcal{O}_{S_0})$ by $\mathfrak{p} \in \text{Proj}S \mapsto f(\mathfrak{p}) = \mathfrak{q} = \mathfrak{p} \cap S_0$, and for basic open set $D(a) \subset \text{Spec}S_0$, define $f^{\sharp}_{D(a)} : \mathcal{O}_{S_0}(D(a)) = (S_0)_a \longrightarrow \mathcal{O}(f^{-1}(D(a)))$ by:

$$\begin{aligned} f^{\sharp}_{D(a)} :&(S_0)_a \longrightarrow \mathcal{O}(f^{-1}(D(a))) \\ b/a^m &\mapsto \begin{pmatrix} f^{\sharp}_{D(a)}(b/a^m) : f^{-1}(D(a)) \to \coprod S_{(\mathfrak{p})} \\ \mathfrak{q} \mapsto b/a^m \end{pmatrix}. \end{aligned}$$

This $f^{\sharp}$ induces a ring homomorphism of stalks(local rings):

$$\begin{aligned} (\mathcal{O}_{S_0})_{\mathfrak{q}} = (S_0)_{\mathfrak{q}} &\xrightarrow{f^{\sharp}_{\mathfrak{q}}} \mathcal{O}_{\mathfrak{p}} = S_{(\mathfrak{p})} \\ b/f &\mapsto b/f. \end{aligned}$$

Since, $\mathfrak{q} = \mathfrak{p} \cap S_0$, this $f^{\sharp}_{\mathfrak{q}}$ is a local ring homomorphism. Hence, we conclude that $(f, f^{\sharp}) : (\text{Proj}S, \mathcal{O}) \longrightarrow (\text{Spec}S_0, \mathcal{O}_{S_0})$ is a natural morphism of schemes.

**Problem20**.
Let $I \subset A$ be a homogeneous ideal, and let $I_d \subset I$ be the set of all homogeneous elements in $I$ having degree $d$. Then, we have

$$I = \bigoplus_{d \geq 0} I_d.$$

Furthermore, any ideal $I$ satisfying this property is homogeneous. To show that $\bar{I}$ is homogeneous, consider $a \in \bar{I}$. By definition of $\bar{I}$, there is $m \geq 0$ such that $at_i^m \in I$ for all $0 \leq i \leq n$. Write $a = \sum_{d \geq 0} a_d$, where $a_d$ is homogeneous element of degree $d$. Then, $a_d t_i^m \in I$ since $at_i^m = \sum_{d \geq 0} a_d t_i^m \in I$ with $a_d t_i^m$ having degree $d + m$, homogeneous. Since this holds for every $0 \leq i \leq n$, we see that $a_d \in \bar{I}$ for every $d \geq 0$. Hence, $\bar{I}$ is homogeneous.

**Problem21**.
Define $V(I) = \{\mathfrak{p} \in \text{Proj}A | I \subset \mathfrak{p}\}$.

$\Rightarrow$) It suffices to show that $I$ and $\bar{I}$ define the same closed subschemes of Proj$A$. First, we show the set-theoretic equality $V(I) = V(\bar{I})$. It is clear that $V(I) \supset V(\bar{I})$, since $I \subset \bar{I}$. Let $\mathfrak{p} \in V(I)$ and $a \in \bar{I}$. We claim that $a \in \mathfrak{p}$. By definition of $\bar{I}$, we have some $m \geq 0$ such that $at_i^m \in I$ for all $0 \leq i \leq n$. Suppose $a \notin \mathfrak{p}$, then we must have $t_i^m \in \mathfrak{p}$ for all $i$. This forces that the ideal $(t_0, \cdots, t_n)$ is contained in $\mathfrak{p}$. Since $\mathfrak{p} \in \text{Proj}A$, $\mathfrak{p}$ cannot contain $(t_0, \cdots, t_n)$. Thus, we proved our claim, namely, $V(I) \subset V(\bar{I})$. Hence, $V(I) = V(\bar{I})$ follows.

Now, we need to show that $V(I) \simeq \mathrm{Proj}(A/I)$ and $V(\bar{I}) \simeq \mathrm{Proj}(A/\bar{I})$ are isomorphic as schemes. For, we consider the canonical surjection $A/I \longrightarrow A/\bar{I}$ given by $\bar{a} \mapsto \hat{a}$. This induces a surjection of localized rings $(A/I)_{(f)} \longrightarrow (A/\bar{I})_{(f)}$ which associates $\bar{a}/f^r$ to $\hat{a}/f^r$ for homogeneous $a, f$ with $\deg f > 0$ and $\deg a = r\deg f$. It will be enough to show that this map is also injective. But if $\hat{a}/f^r = 0$, then $f^m a \in \bar{I}$. There is an integer $N$ such that $t_0^N f^m a, \cdots, t_n^N f^m a \in I$ and for $k$ large enough $f^k a \in I$, so $\bar{a}/f^r = 0$ in $(A/I)_{(f)}$.

$\Leftarrow$) Suppose $I, J$ define the same closed subschemes of $\mathrm{Proj}A$. Then, we have $(A/I)_{(f)} \simeq (A/J)_{(f)}$ for any homogeneous element $f \in A^+$ via $\bar{a}/f^r \mapsto \hat{a}/f^r$. By the way, $a \in \bar{J}$ if and only if there is $m$ such that $at_i^m \in J$ for all $i$. This means that $\hat{a} = 0$ in $(A/J)_{(t_i)}$ for each $i$. By the isomorphism, we have $\bar{a} = 0$ in $(A/I)_{(t_i)}$ for each $i$. Again, this is equivalent to $at_i^{m_i} \in I$ for some $m_i$. Taking maximum of $m_i$, we obtain that $a \in \bar{I}$. Hence, $a \in \bar{J} \Leftrightarrow a \in \bar{I}$, giving that $\bar{I} = \bar{J}$.