More on the crossing number of $K_n$: Monotone drawings

Bernardo M. Ábrego a,1 Oswin Aichholzer b,2
Silvia Fernández-Merchant a,3 Pedro Ramos c,4
Gelasio Salazar d,5

a California State University, Northridge.
b Graz University of Technology
c Universidad de Alcalá
d Universidad Autónoma de San Luis Potosí

Abstract
The Harary-Hill conjecture states that the minimum number of crossings in a drawing of the complete graph $K_n$ is $Z(n) := \frac{1}{4}
\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$. This conjecture was recently proved for 2-page book drawings of $K_n$. As an extension of this technique, we prove the conjecture for monotone drawings of $K_n$, that is, drawings where all vertices have different $x$-coordinates and the edges are $x$-monotone curves.

Keywords: Crossing number, Topological drawing, Complete graph, Monotone drawing, k-edge.

1 Email: bernardo.abrego@csun.edu
2 Partially supported by the ESF EUROCORES programme EuroGIGA, CRP ComPoSe, under grant FWF I648-N18. Email: oaich@ist.tugraz.at
3 Email: silvia.fernandez@csun.edu
4 Partially supported by MEC grant MTM2011-22792 and ESF EUROCORES programme EuroGIGA, CRP ComPoSe, grant EUI-EURC-2011-4306. Email: pedro.ramos@uah.es
5 Supported by CONACYT grant 106432. Email: gsalazar@ifisica.uaslp.mx
1 Introduction

We consider drawings $D$ of the complete graph $K_n$ in the plane. A drawing $D$ is monotone if all its vertices have different $x$-coordinates and its edges are $x$-monotone simple curves. $D$ is a 2-page book drawing if all its vertices are on a line $\ell$ and each edge is fully contained in one of the two half-planes defined by $\ell$. The number of (pairwise) crossings in $D$ is denoted by $\text{cr}(D)$. The crossing number of $K_n$, denoted by $\text{cr}(K_n)$, is the minimum of $\text{cr}(D)$ over all drawings $D$ of $K_n$. The Harary-Hill conjecture [6,7] states that

$$\text{cr}(K_n) = Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor.$$ 

For more on the history of this problem see [4]. There are drawings of $K_n$ with exactly $Z(n)$ crossings. In fact, there are monotone and 2-page book drawings of $K_n$ with exactly $Z(n)$ crossings [1,5]. So it is natural to conjecture that the crossing number of $K_n$ restricted to each of these classes of drawings is also $Z(n)$. The conjecture for 2-page book drawings of $K_n$ was recently proved by the authors [1,2]. Here we prove the conjecture for the monotone case. More precisely, let mon-cr($K_n$) be the minimum of $\text{cr}(D)$ taken over all monotone drawings $D$ of $K_n$. (For more on the monotone crossing number of a graph see [8].)

**Theorem 1.1** For every positive integer $n$, $\text{mon-cr}(K_n) = Z(n)$.

Since any 2-page book drawing of $K_n$ can be drawn as a monotone drawing preserving the number of crossings, Theorem 1.1 implies Theorem 10 in [1].

2 Proof of Theorem 1.1

As usual, it is enough to only consider good drawings of $K_n$ (two incident edges do not intersect and any two edges intersect each other at most once) as the typical transformations to reduce to this case preserve monotonicity. Our proof is a natural extension of that presented in [1] for 2-page book drawings of $K_n$. We need the following definitions and results from [1]. In a good drawing of $K_n$, the three edges connecting any three vertices $x, y, z$ do not intersect each other, forming a closed simple curve, a triangle. The triangle $xyz$ is positive if it is oriented counterclockwise and negative otherwise. The edge $xy$ is a $k$-edge if exactly $k$ of the triangles $xyz$ are positive or exactly $k$ are negative. For a drawing $D$ of $K_n$, let $E_k(D)$ be the number of $k$-edges of
and
\[ E_{\leq k}(D) = \sum_{j=0}^{k} (k + 1 - j) E_j(D). \] (1)

**Theorem 2.1** [1] For any good drawing \( D \) of \( K_n \) in the plane,
\[ \text{cr}(D) = 2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor - \frac{1}{2} (1 + (-1)^n) E_{\leq \lfloor n/2 \rfloor - 2}(D). \]

**Theorem 2.2** If \( n \geq 3 \), \( 0 \leq k < n/2 - 1 \), and \( D \) a monotone drawing of \( K_n \), then
\[ E_{\leq k}(D) \geq 3 \binom{k+3}{3}. \]

**Proof.** We proceed by induction on \( n \). The case \( n = 3 \) holds trivially. For \( n \geq 4 \), consider a monotone drawing \( D \) of \( K_n \) with vertices, from left to right, \( 1, 2, \ldots, n \). Remove the point \( n \) and all incident edges to obtain a monotone drawing \( D' \) of \( K_{n-1} \). A \( k \)-edge in \( D \) is invariant if it is also a \( k \)-edge in \( D' \).

Let \( E_{\leq k}(D, D') \) be the number of invariant \( \leq k \)-edges. We compare (1) to
\[ E_{\leq k-1}(D') = \sum_{j=0}^{k-1} (k - j) E_j(D'). \] (2)

As shown in [1], for \( j \leq k \) a \( j \)-edge incident to \( n \) contributes \( k - j \) to (1) and nothing to (2), an invariant \( \leq k \)-edge contributes 1 more to (1) than to (2), and all other edges contribute the same to (1) and (2). Therefore
\[ E_{\leq k}(D) = E_{\leq k-1}(D') + \sum_{i=0}^{k} (k + 1 - j)e_j(n) + E_{\leq k}(D, D'), \] (3)

where \( e_j(n) \) is the number of \( j \)-edges incident to \( n \) in \( D \). We show that \( e_j(n) = 2 \) for each \( 0 \leq j \leq k \), and that \( E_{\leq k}(D, D') \geq \binom{k+2}{2} \). This, the induction hypothesis \( E_{\leq k-1}(D') \geq 3 \binom{k+2}{3} \), and (3) imply the result.

To prove that \( e_j(n) = 2 \), we order the edges incident to \( n \) from top to bottom according to the order in which they leave \( n \). Note that the \( i'th \) edge, say \( vn \) is an \((i-1)\)-edge as it forms positive triangles with all edges above it from \( n \) and negative triangles with all other edges from \( n \). We now prove that \( E_{\leq k}(D, D') \geq \binom{k+2}{2} \). In fact, we prove that for each \( 1 \leq j \leq k+1 \) there are at least \( k+2-j \) invariant \( \leq k \)-edges incident to \( j \). Label the vertices to the right of \( j \) by \( v_1, v_2, \ldots, v_{n-j} \) according to the order in which they leave \( j \) (from top
to bottom) and suppose $n = v_m$. If the triangle $jv_i x$ is positive, then $x < j$ or $x = v_l$ for some $l < i$. Hence there are at most $j - 1 + i - 1 = j + i - 2$ positive triangles $jv_i x$. Thus $jv_i$ is a $\leq k$-edge whenever $i \leq k + 2 - j$ and it is invariant if $i < m$ (so that triangle $jv_i n = jv_i v_m$ is negative). Similarly, the $i^{th}$ edge from the bottom $jv_{n-j+1-i}$ is an invariant $\leq k$-edge if $i \leq k + 2 - j$ and $n - j + 1 - i > m$. Because $k + 2 - j \leq \lceil \frac{1}{2} n - j - 1 \rceil$ for $k < n/2 - 1$, then there must be at least $k + 2 - j$ edges $jv_i$ above $jn$ or at least $k + 2 - j$ below $jn$ near $j$. Those $k + 2 - j$ edges are invariant $\leq k$-edges.

Theorems 2.1 and 2.2 imply $\text{mon-cr}(K_n) \geq Z(n)$ and thus Theorem 1.1. The authors just learned of the recent work by Balko, Fulek, and Kynčl [3], who also adapted our proof in [1] to the monotone case.

References


