More on the crossing number of K_n: Monotone drawings

Bernardo M. Ábrego [a,1] Oswin Aichholzer [b,2]
Silvia Fernández-Merchant [a,3] Pedro Ramos [c,4]
Gelasio Salazar [d,5]

[a] *California State University, Northridge.*

[b] *Graz University of Technology*

[c] *Universidad de Alcalá*

[d] *Universidad Autónoma de San Luis Potosí*

Abstract

The Harary-Hill conjecture states that the minimum number of crossings in a drawing of the complete graph K_n is $Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$. This conjecture was recently proved for *2-page book drawings* of K_n. As an extension of this technique, we prove the conjecture for *monotone drawings* of K_n, that is, drawings where all vertices have different x-coordinates and the edges are x-monotone curves.

Keywords: Crossing number, Topological drawing, Complete graph, Monotone drawing, k-edge.

[1] Email: bernardo.abrego@csun.edu
[2] Partially supported by the ESF EUROCORES programme EuroGIGA, CRP ComPoSe, under grant FWF I648-N18. Email: oaich@ist.tugraz.at

[3] Email: silvia.fernandez@csun.edu
[4] Partially supported by MEC grant MTM2011-22792 and ESF EUROCORES programme EuroGIGA, CRP ComPoSe, grant EUI-EURC-2011-4306. Email: pedro.ramos@uah.es
[5] Supported by CONACYT grant 106432. Email: gsalazar@ifisica.uaslp.mx

1 Introduction

We consider drawings D of the complete graph K_n in the plane. A drawing D is *monotone* if all its vertices have different x-coordinates and its edges are x-monotone simple curves. D is a *2-page book drawing* if all its vertices are on a line ℓ and each edge is fully contained in one of the two half-planes defined by ℓ. The number of (pairwise) crossings in D is denoted by $\mathrm{cr}(D)$. The *crossing number of* K_n, denoted by $\mathrm{cr}(K_n)$, is the minimum of $\mathrm{cr}(D)$ over all drawings D of K_n. The Harary-Hill conjecture [6,7] states that

$$\mathrm{cr}(K_n) = Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

For more on the history of this problem see [4]. There are drawings of K_n with exactly $Z(n)$ crossings. In fact, there are monotone and 2-page book drawings of K_n with exactly $Z(n)$ crossings [1,5]. So it is natural to conjecture that the crossing number of K_n restricted to each of these classes of drawings is also $Z(n)$. The conjecture for 2-page book drawings of K_n was recently proved by the authors [1,2]. Here we prove the conjecture for the monotone case. More precisely, let $\text{mon-cr}(K_n)$ be the minimum of $\mathrm{cr}(D)$ taken over all monotone drawings D of K_n. (For more on the monotone crossing number of a graph see [8].)

Theorem 1.1 *For every positive integer* n*,* $\text{mon-cr}(K_n) = Z(n)$.

Since any 2-page book drawing of K_n can be drawn as a monotone drawing preserving the number of crossings, Theorem 1.1 implies Theorem 10 in [1].

2 Proof of Theorem 1.1

As usual, it is enough to only consider good drawings of K_n (two incident edges do not intersect and any two edges intersect each other at most once) as the typical transformations to reduce to this case preserve monotonicity. Our proof is a natural extension of that presented in [1] for 2-page book drawings of K_n. We need the following definitions and results from [1]. In a good drawing of K_n, the three edges connecting any three vertices x, y, z do not intersect each other, forming a closed simple curve, a *triangle*. The triangle xyz is *positive* if it is oriented counterclockwise and *negative* otherwise. The edge xy is a k*-edge* if exactly k of the triangles xyz are positive or exactly k are negative. For a drawing D of K_n, let $E_k(D)$ be the number of k-edges of

D and

$$E_{\leq\leq k}\left(D\right)=\sum_{j=0}^{k}\left(k+1-j\right)E_{j}\left(D\right). \tag{1}$$

Theorem 2.1 [1] *For any good drawing D of K_n in the plane,*

$$\operatorname{cr}(D)=2\sum_{k=0}^{\lfloor n/2\rfloor-2}E_{\leq\leq k}(D)-\frac{1}{2}\binom{n}{2}\left\lfloor\frac{n-2}{2}\right\rfloor-\frac{1}{2}(1+(-1)^{n})E_{\leq\leq\lfloor n/2\rfloor-2}(D).$$

Theorem 2.2 *If $n\geq 3$, $0\leq k<n/2-1$, and D a monotone drawing of K_n, then*

$$E_{\leq\leq k}(D)\geq 3\tbinom{k+3}{3}.$$

Proof. We proceed by induction on n. The case $n=3$ holds trivially. For $n\geq 4$, consider a monotone drawing D of K_n with vertices, from left to right, $1,2,\ldots,n$. Remove the point n and all incident edges to obtain a monotone drawing D' of K_{n-1}. A k-edge in D is *invariant* if it is also a k-edge in D'. Let $E_{\leq k}(D,D')$ be the number of invariant $\leq k$-edges. We compare (1) to

$$E_{\leq\leq k-1}\left(D'\right)=\sum_{j=0}^{k-1}\left(k-j\right)E_{j}\left(D'\right). \tag{2}$$

As shown in [1], for $j\leq k$ a j-edge incident to n contributes $k-j$ to (1) and nothing to (2), an invariant $\leq k$-edge contributes 1 more to (1) than to (2), and all other edges contribute the same to (1) and (2). Therefore

$$E_{\leq\leq k}(D)=E_{\leq\leq k-1}(D')+\sum_{i=0}^{k}(k+1-j)e_{j}(n)+E_{\leq k}(D,D'), \tag{3}$$

where $e_j(n)$ is the number of j-edges incident to n in D. We show that $e_j(n)=2$ for each $0\leq j\leq k$, and that $E_{\leq k}(D,D')\geq\binom{k+2}{2}$. This, the induction hypothesis $E_{\leq\leq k-1}(D')\geq 3\binom{k+2}{3}$, and (3) imply the result.

To prove that $e_j(n)=2$, we order the edges incident to n from top to bottom according to the order in which they leave n. Note that the i^th edge, say vn is an $(i-1)$-edge as it forms positive triangles with all edges above it from n and negative triangles with all other edges from n. We now prove that $E_{\leq k}(D,D')\geq\binom{k+2}{2}$. In fact, we prove that for each $1\leq j\leq k+1$ there are at least $k+2-j$ invariant $\leq k$-edges incident to j. Label the vertices to the right of j by $v_1,v_2,\ldots,v_{n-j}$ according to the order in which they leave j (from top to bottom) and suppose $n = v_m$. If the triangle jv_ix is positive, then $x < j$ or $x = v_l$ for some $l < i$. Hence there are at most $j - 1 + i - 1 = j + i - 2$ positive triangles jv_ix. Thus jv_i is a $\leq k$-edge whenever $i \leq k + 2 - j$ and it is invariant if $i < m$ (so that triangle $jv_in = jv_iv_m$ is negative). Similarly, the i^{th} edge from the bottom $jv_{n-j+1-i}$ is an invariant $\leq k$-edge if $i \leq k + 2 - j$ and $n - j + 1 - i > m$. Because $k + 2 - j \leq \lceil \frac{1}{2}n - j - 1 \rceil$ for $k < n/2 - 1$, then there must be at least $k + 2 - j$ edges jv_i above jn or at least $k + 2 - j$ below jn near j. Those $k + 2 - j$ edges are invariant $\leq k$-edges. $\square$

Theorems 2.1 and 2.2 imply mon-cr$(K_n) \geq Z(n)$ and thus Theorem 1.1. The authors just learned of the recent work by Balko, Fulek, and Kynčl [3], who also adapted our proof in [1] to the monotone case.

References

[1] B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos, and G. Salazar, The 2-page crossing number of K_n. To appear in *Disc. Comp. Geom.* (2013) arXiv:1206.5669.

[2] B. M. Ábrego, O. Aichholzer, S. Fernández-Merchant, P. Ramos, and G. Salazar, The 2-page crossing number of K_n. In *28th Ann. ACM Symp. Comp. Geom.*, pp. 397–403, Chapel Hill, NC, USA, 2012.

[3] M. Balko, R. Fulek, and J. Kynčl, Monotone Crossing Number of Complete Graphs, Manuscript (2013).

[4] L. Beineke and R. Wilson. The early history of the brick factory problem. *Math. Intelligencer*, 32 (2010) 41–48.

[5] J. Blažek and M. Koman. A minimal problem concerning complete plane graphs. In M. Fiedler, editor, *Theory of graphs and its applications*, pp. 113–117. Czech. Acad. of Sci., 1964.

[6] R. K. Guy. A combinatorial problem. *Bull. Malayan Math. Soc.*, 7 (1960) 68–72.

[7] F. Harary and A. Hill. On the number of crossings in a complete graph. *Proc. Edinburgh Math. Soc.*, 13 (1963) 333–338.

[8] J. Pach and G. Tóth. Monotone crossing number. *Graph drawing*, 278-289, Lecture Notes in Comput. Sci., 7034, Springer, Heidelberg, 2012.