A Lower Bound for the Rectilinear Crossing Number.

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Abstract

We give a new lower bound for the rectilinear crossing number $\overline{cr}(n)$ of the complete geometric graph K_n. We prove that $\overline{cr}(n) \geq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$ and we extend the proof of the result to pseudolinear drawings of K_n.

1 Introduction

The *crossing number* $cr(G)$ of a simple graph G is the minimum number of edge crossings in any drawing of G in the plane, where each edge is a simple curve. The *rectilinear crossing number* $\overline{cr}(G)$ is the minimum number of edge crossings when G is drawn in the plane using straight segments as edges. The crossing numbers have many applications to Discrete Geometry and Computer Science, see for example [7] and [9].

In this paper we study the problem of determining $\overline{cr}(K_n)$, where K_n denotes the complete graph on n vertices. For simplicity we write $\overline{cr}(n) = \overline{cr}(K_n)$. An equivalent formulation of the problem is to find the minimum number of convex quadrilaterals determined by n points in general position (no three points on a line).

We mention here that $cr(K_n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$ was conjectured by Zarankiewicz [12] and Guy [3], and there are (non-rectilinear) drawings of K_n achieving this number. Of course $cr(K_n) \leq \overline{cr}(K_n)$ but from the exact values of $\overline{cr}(n)$ for $n \leq 12$ [1], it is known that $cr(K_8) < \overline{cr}(K_8)$.

Jensen [6] and Singer [10] were the first to settle $\overline{cr}(n) = \Theta(n^4)$. In fact, since $\overline{cr}(5) = 1$ then by an averaging argument it is easy to deduce that $\overline{cr}(n) \geq \frac{1}{5}\binom{n}{4}$. This same idea was used by Brodsky et al [2] when they obtained $\overline{cr}(10) = 62$, to deduce $\overline{cr}(n) \geq 0.3001\binom{n}{4}$. Later Aicholzer et al [1] calculated $\overline{cr}(12) = 153$ and used this to get $\overline{cr}(n) \geq 0.3115\binom{n}{4}$. Very recently Wagner [11], following different methods proved $\overline{cr}(n) \geq 0.3288\binom{n}{4}$. On the other hand Brodsky et al [2] constructed rectilinear drawings of K_n showing $\overline{cr}(n) \leq \frac{6467}{16848}\binom{n}{4} \leq 0.3838\binom{n}{4}$. In this paper we prove the following theorem which gives as a lower bound for $\overline{cr}(n)$ the exact value conjectured by Zarankiewicz and Guy for $cr(K_n)$.

Theorem 1 $\overline{cr}(n) \geq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$.

It is known that $c^* = \lim_{n\to\infty} \overline{cr}(n)/\binom{n}{4} > 0$ exists. Our theorem gives $c^* \geq 3/8 = 0.375$ and it can in fact be generalized to a larger class of drawings of K_n. Namely, those obtained from the concept of *simple allowable sequences of permutations* introduced by Goodman and Pollack [4]. We denote by $\mathbb{P}^2$ the real projective plane, a *pseudoline* ℓ is a simple closed curve whose removal does not disconnect $\mathbb{P}^2$. A finite set P in the plane is a *generalized configuration* if it consists of a set of points, together with a set of pseudolines joining each pair of points subject to the condition that each pseudoline intersects every other exactly once. If there is a single pseudoline for every pair then the generalized configuration is called *simple*.

Consider a drawing of K_n in the (projective) plane where each edge is represented by a simple curve. If each of these edges can be extended to a pseudoline in such a way that the resulting structure is a simple generalized configuration then we call such a drawing a *pseudolinear drawing* of K_n. We call *pseudosegments* the edges of a pseudolinear drawing. Clearly, every rectilinear drawing of K_n is also pseudolinear. Thus the number $\widetilde{cr}(n)$, defined as the minimum number of edge crossings over all pseudolinear drawings of K_n, generalizes the quantity $\overline{cr}(n)$ and satisfies $\widetilde{cr}(n) \leq \overline{cr}(n)$. In this context we prove the following stronger result.

Theorem 2 $\widetilde{cr}(n) \geq \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$.

If a pseudolinear drawing is combinatorially equivalent to a rectilinear drawing then it is called *stretchable*. It is known that almost all pseudolinear drawings are non-stretchable. So it is conceivable that $\widetilde{cr}(n) < \overline{cr}(n)$ for n sufficiently large, but at the moment we have no other evidence to support this. We also mention that the problem of determining whether a pseudolinear drawing is stretchable is NP-hard [8].

2 Allowable Sequences

Given a set P of n points in the plane, no three of them collinear, we construct the $\left(\binom{n}{2}+1\right) \times n$ matrix $S(P)$ as follows.

Consider any circle C containing P in its interior. Let ℓ be the vertical right-hand side tangent line to C. We can assume without loss of generality that no segment in P is perpendicular to ℓ, we can also assume that no two segments in P are parallel, otherwise we can perturb the set P without changing the structure of its crossings. Label the points of P from 1 to n according to the order of their projections to ℓ, 1 being the lowest and n the highest. For each segment $\overline{ij}$ in P, let $c_{ij} = c_{ji}$ be the point in the upper half of C such that the tangent line to C at c_{ij} is perpendicular to $\overline{ij}$. This gives a linear order on the segments of P, inherited from the counter-clockwise order of the points c_{ij} in C. Denote by t_r the r^{th} pair of points (segment) in P under this order. Indistinctly we use t_r to denote an unordered pair $\{i, j\}$ or the point $c_{ij} = c_{ji}$. Using this, we recursively construct the matrix $S(P)$. The first row is $(1, 2, ..., n)$, and the $(k+1)^{th}$ row is obtained from the k^{th} row by switching the pair t_k. $S(P)$ is half a period of what is commonly referred as a *circular sequence of permutations* of P [4].

$S(P)$ satisfies the following properties.

1. The first row of $S(P)$ is the n-tuple $(1, 2, 3, ..., n)$, the last row of $S(P)$ is the n-tuple $(n, n-1, ..., 2, 1)$, and any row of $S(P)$ is a permutation of its first row.

2. Any row $r \geq 2$ is obtained from the previous row by switching two consecutive entries of the row $r-1$.

3. If the r^{th} row is obtained by switching the entries $S_{r-1,c}$ and $S_{r-1,c+1}$ in the $(r-1)^{th}$ row then $S_{r-1,c} < S_{r-1,c+1}$.

4. For every $1 \leq i < j \leq n$ there exists a unique row $1 \leq r \leq \binom{n}{2}$ such that the entries i and j are switched from row r to row $r+1$, i.e., $t_r = \{i, j\}$, $S_{r,c} = i < j < S_{r,c+1}$, and $S_{r+1,c} = j > i = S_{r+1,c+1}$ for some $1 \leq c \leq n-1$.

A simple allowable sequence of permutations is a combinatorial abstraction of a circular sequence of permutations associated with a configuration of points. It is defined as a doubly infinite periodic sequence of permutations of $1, 2, \ldots, n$ satisfying that every permutation is obtained from the previous one by switching two adjacent numbers, and after i and j have been switched they do not switch again until all other pairs have switched. For the purposes of this paper we only use half a period of an allowable sequence. This translates to any $\left(\binom{n}{2}+1\right) \times n$ matrix $S(P)$ satisfying properties

1-4. From now on $S(P)$ will be such a matrix, not necessarily obtained as the circular sequence of permutations of a point set P.

It was proved by Goodman and Pollack [5] that every simple allowable sequence of permutations can be realized by a generalized configuration of points where the matrix $S(P)$ is determined by the cyclic order in which the connecting pseudolines meet a distinguished pseudoline (for example the pseudoline at infinity).

Next we establish when two pseudosegments do not intersect by means of the matrix $S(P)$. Given a simple generalized configuration of points P, we say that two pseudosegments $\widetilde{ab}$ and $\widetilde{cd}$ are *separated* if there exists a pseudoline in P that leaves $\widetilde{ab}$ and $\widetilde{cd}$ in different sides. Note that any two *non-incident* pseudosegments (i.e., they do not share endpoints), either intersect in their interior (generate a crossing) or are separated. Thus $\widetilde{cr}(G_P) = \widetilde{cr}(P)$ is the number of non-incident pairs of pseudosegments minus the number of separated pseudosegments, where G_P is a pseudolinear drawing of K_n associated to $S(P)$.

Let $<_r$ be the linear order on $\{1, 2, 3, \ldots, n\}$ induced by the r^{th} row of $S(P)$. Observe that $\widetilde{ab}$ and $\widetilde{cd}$ are separated if and only if there is a row r such that $a, b <_r c, d$ or $c, d <_r a, b$. In this case we say $\widetilde{ab}$ and $\widetilde{cd}$ are *separated in row r*.

Lemma 3 allows us to count the number of separated pseudosegments in P. We say $\widetilde{ab}$ and $\widetilde{cd}$ are *neighbors in row r* if they are separated in row r but not in row $r-1$.

Lemma 3 *$\widetilde{ab}$ and $\widetilde{cd}$ are separated if and only if there is a unique row r where $\widetilde{ab}$ and $\widetilde{cd}$ are neighbors.*

Proof. First note that if $\widetilde{ab}$ and $\widetilde{cd}$ are neighbors, then they are separated by definition. Now assume $\widetilde{ab}$ and $\widetilde{cd}$ are separated, and let R be the last row where they are separated. If $\widetilde{ab}$ and $\widetilde{cd}$ are separated in all rows above R then they are separated in the first and consequently in the last rows, that is $R = \binom{n}{2} + 1$. This is impossible since having $\widetilde{ab}$ and $\widetilde{cd}$ separated in every row implies that they never reversed their order.

Consider the largest row $r \leq R$ such that $\widetilde{ab}$ and $\widetilde{cd}$ are not separated in row $r-1$. Then $\widetilde{ab}$ and $\widetilde{cd}$ are neighbors in row r. Finally, to prove that such a row is unique, let $r_0 < r_1$ be two rows where $\widetilde{ab}$ and $\widetilde{cd}$ are neighbors. Assume without loss of generality that $a <_{r_0} b <_{r_0} c <_{r_0} d$. Then $a <_{r_0-1} c <_{r_0-1} b <_{r_0-1} d$ and, since b and c switch exactly once, $b <_{r_1} c$. Also, by definition, one of the pairs $\widetilde{ac}, \widetilde{ad}$, or $\widetilde{bd}$ switches from row $r_1 - 1$ to row r_1. Since such a pair switches exactly once, then it has opposite orders in rows r_0 and r_1. Therefore one of the following should be satisfied

$$b <_{r_1} c <_{r_1} a <_{r_1} d, \text{ or } b <_{r_1} d <_{r_1} a <_{r_1} c, \text{ or } a <_{r_1} d <_{r_1} b <_{r_1} c,$$

but then $\widetilde{ab}$ and $\widetilde{cd}$ are not separated in row r_1. ■

For all $i \neq j$ in P, write $f_P\left(\widetilde{ij}\right) = (r, c)$, if i and j switch in row r and column c, that is $S_{r,c} = i = S_{r+1,c+1}$ and $S_{r,c+1} = j = S_{r+1,c}$. Note that this is well defined since the relative order of each pair of points $\{i, j\}$ in P is changed exactly once.

For $1 \leq c \leq n-1$ define

$$C_P(c) = \left\{ r : \text{there exist } i, j \text{ such that } f_P\left(\widetilde{ij}\right) = (r, c) \right\},$$

and let $ch_P(c) = ch(c) = |C_P(c)|$. In other words denotes the number of changes (switches) in column c.

Lemma 4 *For any simple generalized configuration P of n points in the plane*

$$\widetilde{cr}(P) = 3\binom{n}{4} - \sum_{j=1}^{n-1} (j-1)(n-1-j)\, ch(j).$$

Proof. Since each four points in P determine three pairs of non-incident pseudosegments, there are $3\binom{n}{4}$ pairs of non-incident pseudosegments in P. It remains to prove that $\sum_{j=1}^{n-1}(j-1)(n-1-j)\,ch(j)$ of these pairs are separated (non-crossing). Note that $\widetilde{ab}$ and $\widetilde{cd}$ are neighbors in row r if and only if there are $x \in \{a,b\}$, $y \in \{c,d\}$ such that x and y switch from row $r-1$ to row r. By Lemma 3, if $t_r = \{i,j\}$ and $i<j$ then all pairs $\widetilde{hj}$ and $\widetilde{ik}$ are neighbors (in row r) whenever $h <_r j$ and $i <_r k$. If $f_P\left(\widetilde{ij}\right) = (r,c)$ then row r accounts for $(c-1)(n-1-c)$ neighboring pairs of pseudosegments. Moreover, Lemma 3 guarantees that, when adding these quantities over all rows, we are counting all separated pairs of pseudosegments exactly once. ■

3 Proof of Theorem 2

Note that for fixed $1 \le i \le \binom{n}{2}$, i switches exactly once with each number $j \ne i$, that is

$$\left|\left\{f_P\left(\widetilde{ij}\right) : 1 \le j \le n, j \ne i\right\}\right| = n-1.$$

Moreover, since n is the last entry in row 1 and the first entry in row $\binom{n}{2}+1$, then when $i=n$ these $n-1$ switches occur in different columns, that is

$$\left\{1 \le c \le n-1 : f_P\left(\widetilde{nj}\right) = (r,c) \text{ for some } 1 \le r \le \binom{n}{2}, \text{ and } 1 \le j < n\right\} = \{1,2,...,n-1\}.$$

Therefore we can define $R_P(c) = r$ to be the unique row r where the change of n in column c occurs, i.e., there exists $1 \le j < n$ such that $f_P\left(\widetilde{nj}\right) = (r,c)$. Also for $1 \le c \le n-1$ define the number of changes in column c above and below row $R_P(c)$ as

$$\begin{aligned} A_P(c) &= \left\{r < R_P(c) : \text{there exist } i,j \text{ such that } f_P\left(\widetilde{ij}\right) = (r,c)\right\} \\ B_P(c) &= \left\{r > R_P(c) : \text{there exist } i,j \text{ such that } f_P\left(\widetilde{ij}\right) = (r,c)\right\}. \end{aligned}$$

The proof of the Theorem is based on the identity from Lemma 4, together with the next two lemmas. Let $m = \lfloor n/2 \rfloor$

Lemma 5 *For any simple generalized configuration P of n points in the plane and $1 \le k \le m-1$ we have*

$$|A_P(k)| + |B_P(n-k)| \ge k.$$

Proof. For $1 \le j \le k$ let

$$\begin{aligned} g(j) &= \min\left\{r : \text{there exists } i \text{ such that } f_P\left(\widetilde{ij}\right) = (r,k)\right\} \\ h(j) &= \min\left\{r : \text{there exists } i \text{ such that } f_P\left(\widetilde{ij}\right) = (r,n-k)\right\}. \end{aligned}$$

Since all $g(1), g(2), ..., g(k), h(1), h(2), ..., h(k)$ are different, and $A_P(k)$ and $B_P(n-k)$ are disjoint, then it is enough to prove that for all $1 \le j \le k$, either $h(j) \in B_P(n-k)$ or $g(j) \in A_P(k)$.

Assume that $h(j) \notin B_P(n-k)$. Then, since $h(j) \ne R_P(n-k)$, $h(j) < R_P(n-k)$. Observe that $g(j) < h(j)$ and $R_P(n-k) < R_P(k)$ then

$$g(j) < h(j) < R_P(n-k) < R_P(k).$$

Therefore $g(j) \in A_P(k)$. ■

Lemma 6 *For any simple generalized configuration P of n points in the plane and $1 \leq k \leq m-1$ we have*

$$\sum_{c=1}^{k} \left(ch_P(c) + ch_P(n-c) \right) \geq 3\left(1+2+3+...+k\right) = 3\binom{k+1}{2}.$$

Proof. By induction on $|P| = n$. The statement is true for $|P| = 3$ by vacuity.

Consider the matrix $S(P)$ and let $P' = P - \{n\}$. Note that $S(P')$ is the matrix obtained from erasing the unique entry equal to n in each row of $S(P)$ and shifting one column left the necessary elements of $S(P)$. Also the rows where the corresponding change involves n are deleted.

Note that for $1 \leq c \leq n-2$

$$C_{P'}(c) = A_P(c) \cup B_P(c+1).$$

Thus for $1 \leq c \leq n-2$

$$ch_{P'}(c) = |A_P(c)| + |B_P(c+1)|. \tag{1}$$

Also notice that

$$B_P(1) = A_P(n-1) = \varnothing. \tag{2}$$

and for $1 \leq c \leq n-1$

$$ch_P(c) = |A_P(c)| + |B_P(c)| + 1. \tag{3}$$

Then by definition and (3)

$$\begin{aligned} \sum_{c=1}^{k} \left(ch_P(c) + ch_P(n-c) \right) &= \sum_{c=1}^{k} \left(|A_P(c)| + |B_P(c)| + |A_P(n-c)| + |B_P(n-c)| + 2 \right) \\ &= 2k + \sum_{c=1}^{k} \left(|A_P(c)| + |B_P(c)| + |A_P(n-c)| + |B_P(n-c)| \right), \end{aligned}$$

separating one term from each sum we get

$$\begin{aligned} \sum_{c=1}^{k} \left(ch_P(c) + ch_P(n-c) \right) &= 2k + |A_P(k)| + |B_P(1)| + \sum_{c=1}^{k-1} \left(|A_P(c)| + |B_P(c+1)| \right) + \\ &\quad + |A_P(n-1)| + |B_P(n-k)| + \sum_{c=2}^{k} \left(|A_P(n-c)| + |B_P(n-c+1)| \right), \end{aligned}$$

then by (1) and (2),

$$\begin{aligned} \sum_{c=1}^{k} \left(ch_P(c) + ch_P(n-c) \right) &= 2k + |A_P(k)| + |B_P(n-k)| + \sum_{c=1}^{k-1} ch_{P'}(c) + \sum_{c=2}^{k} ch_{P'}(n-c) \\ &= 2k + |A_P(k)| + |B_P(n-k)| + \sum_{c=1}^{k-1} \left(ch_{P'}(c) + ch_{P'}(n-1-c) \right). \end{aligned}$$

Finally, by induction and Lemma 5,

$$\begin{aligned} \sum_{c=1}^{k} \left(ch_P(c) + ch_P(n-c) \right) &\geq 2k + k + 3\left(1+2+...+(k-1)\right) \\ &= 3\left(1+2+...+k\right) = 3\binom{k+1}{2}. \end{aligned}$$

■

Proof of Theorem 2. By Lemma 4, it is enough to find an upper bound for the expression

$$\sum_{c=1}^{n-1} (c-1)\,(n-1-c)\,ch_P(c).$$

For $1 \leq j \leq m-1$ let $x_j = ch_P(j) + ch_P(n-j)$, and $x_m = ch_P(m) + ch_P(m+1)$ if n is odd, otherwise $x_m = ch_P(m)$. Under these definitions and according to Lemma 5, together with the fact that $\sum_{j=1}^{m} x_j = \binom{n}{2}$, it is enough to find the maximum of the function

$$f(x_1, x_2, \ldots, x_m) = \sum_{j=1}^{m} (j-1)\,(n-1-j)\,x_j$$

subject to the following linear conditions:

$$\sum_{j=1}^{m} x_j = \binom{n}{2} \text{ and } \sum_{j=1}^{k} x_j \geq 3\binom{k+1}{2} \text{ for every } 1 \leq k \leq m-1.$$

It is easy to see that the maximum occurs if and only if $x_k = 3k$ for all $1 \leq k \leq m-1$ and $x_m = \binom{n}{2} - 3\binom{m}{2}$. If this is the case then

$$f(x_1, x_2, \ldots, x_m) = \begin{cases} \frac{1}{64}\,(n-3)\,(n-1)\left(7n^2-12n-3\right) & \text{if } n \text{ is odd} \\ \frac{1}{64} n\,(n-2)\left(7n^2-26n+16\right) & \text{if } n \text{ is even.} \end{cases}$$

Therefore, by Lemma 5, we conclude that

$$\widetilde{cr}\,(P) \geq \begin{cases} \frac{1}{64}\,(n-3)^2\,(n-1)^2 & \text{if } n \text{ is odd} \\ \frac{1}{64} n\,(n-2)^2\,(n-4) & \text{if } n \text{ is even.} \end{cases}$$

i.e.,

$$\widetilde{cr}\,(P) \geq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

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