Non-Shellable Drawings of K_n with Few Crossings

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Abstract

In the early 60s, Harary and Hill conjectured $H(n) := \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$ to be the minimum number of crossings among all drawings of the complete graph K_n. It has recently been shown that this conjecture holds for so-called shellable drawings of K_n.

For $n \geq 11$ odd, we construct a non-shellable family of drawings of K_n with exactly $H(n)$ crossings. In particular, every edge in our drawings is intersected by at least one other edge. So far only two other families were known to achieve the conjectured minimum of crossings, both of them being shellable.

1 Introduction

The Harary-Hill Conjecture [4] states that the minimum number of crossings in a drawing of the complete graph K_n in the plane is

$$H(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor.$$

Essentially, only two families of drawings of K_n (besides sporadic examples or slight modifications) were known to achieve $H(n)$ crossings: Hill's constructions [4], which are also called cylindrical drawings (see Figure 1), and the Blažek-Koman constructions [3], which are 2-page book drawings (see for example Figure 2).

Very recently, shellability was defined [1] using the cells of a drawing of a graph (an embedding). A drawing D of K_n is *s-shellable* if there exists a sequence $v_1, v_2, \ldots, v_s$ of the vertices and a cell F of D with the following property. For $1 \leq i < j \leq s$, let D_{ij} be the drawing obtained from D by removing $v_1, v_2, \ldots v_{i-1}, v_{j+1}, v_{j+2}, \ldots, v_s$; then for all $1 \leq i < j \leq s$, the vertices v_i and v_j are incident to the cell of D_{ij} that contains F. The sequence $v_1, v_2, \ldots, v_s$ is an *s-shelling* of D. In particular, for $s \geq 2$, an s-shellable drawing of K_n must have a cell with at least the two vertices v_1 and v_s on its boundary.

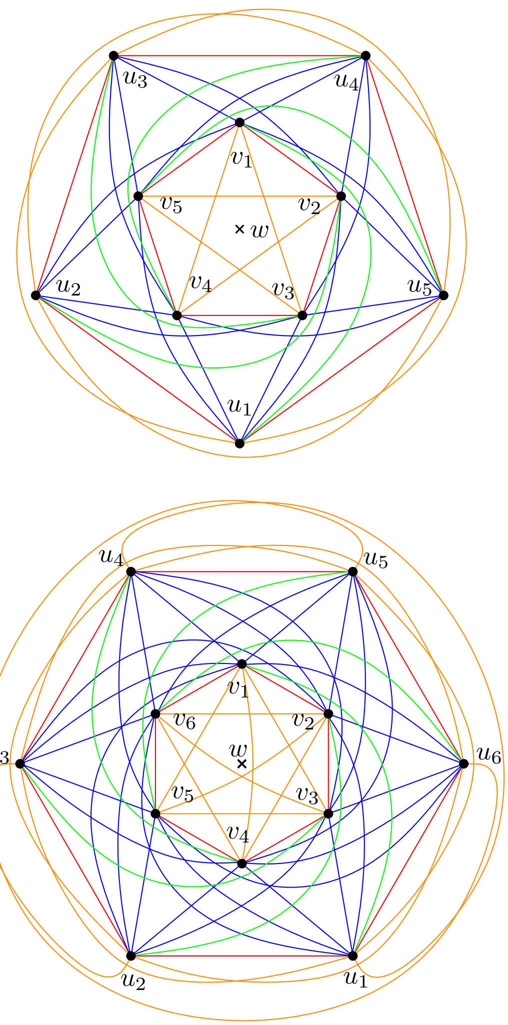

Figure 1: Harary-Hill drawings H_{10} and H_{12}.

Using this concept, in [1] the Harary-Hill Conjecture has been proven when restricted to s-shellable drawings of K_n with $s \geq \lfloor\frac{n}{2}\rfloor$. For simplicity, such sets are called shellable. Moreover, it has been shown that cylindrical drawings are $\lfloor\frac{n}{2}\rfloor$-shellable and 2-page book drawings are n-shellable. Thus, the Harary-Hill Conjecture holds

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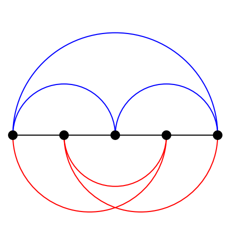

Figure 2: Crossing optimal 2-page book drawings with $n = 5, 6$ vertices.

when restricted to cylindrical drawings and to 2-page book drawings [2].

Shellability can be seen as the first combinatorial classification to identify drawings for which the Harary-Hill conjecture provably holds. A natural question is whether all drawings of K_n with $H(n)$ crossings are s-shellable for some $s \geq \lfloor \frac{n}{2} \rfloor$, or if this is at least the case for graphs of sufficient large cardinality.

In this paper, we make the first attempt to go beyond shellability, and answer both questions in the negative by presenting a new family of drawings of K_n with exactly $H(n)$ crossings that are non-shellable.

In the next section, we first present the basics of one existing family of drawings with $H(n)$ crossings, namely the Harary-Hill cylindrical drawings. Then we give the details of our new constructions and show that it also has $H(n)$ crossings. In Section 3, we present a classification of all essentially different (that is, all weak isomorphism classes of) crossing optimal drawings for small sets and relate them to our general constructions.

2 The new family

For each $m \geq 3$, we construct a drawing $N_{m,m,1}$ of K_{2m+1} with exactly $H(2m+1)$ crossings. These drawings have the an additional property that for $m \geq 5$, every edge of $N_{m,m,1}$ participates in at least one crossing. In other words, each cell determined by $N_{m,m,1}$ has at most one vertex on its boundary, and thus $N_{m,m,1}$ is non-shellable. Moreover, in Section 2.4 we give constructions for non-shellable drawings $N_{m,m,2}$ and $N_{m,m,3}$ for m odd. These drawings have $H(2m+2)$ and $H(2m+3)$ crossings, respectively.

We start by describing in detail the Harary-Hill drawing of K_{2m}.

2.1 The Harary-Hill drawing of K_{2m}

In 1958, Anthony Hill constructed a drawing H_n of K_n, for any positive integer n, with exactly $H(n)$ crossings. We describe in detail the drawing H_{2m}, for $m \geq 3$. First, H_{2m} is a *good drawing*, that is, a drawing in which (i) edges do not self-intersect, (ii) any two incident edges do not cross, (iii) any two vertex-disjoint edges cross at most once, and (iv) all crossings of edges are proper.

The vertices of H_{2m} are drawn as the vertices of two regular convex polygons $U = \{u_1, u_2, \dots, u_m\}$ and $V = \{v_1, v_2, \dots, v_m\}$ (labeled clockwise), and both are concentric with center w. U is larger than V, and they are positioned in such a way that u_1, w, and v_1 are collinear if m is odd, and u_1, w, and the midpoint of $v_1 v_m$ are collinear if m is even (see Figure 1).

The edges joining two vertices of V stay inside (or on the boundary of) V, the edges joining two vertices of U are outside (or on the boundary of) U, and the edges joining vertices of U with vertices of V are outside V and inside U. Finally, for each $1 \leq i \leq n$, the *triangle* $v_i u_i u_{i+1}$ contains V. Here a triangle is the simple cycle that is formed by the three edges $v_i u_i$, $u_i u_{i+1}$, and $u_{i+1} v_i$ (this cycle is not self-intersecting as H_{2m} is a good drawing). All edges can be drawn so that they do not pass through w.

To ease the presentation, we require that the edges $u_i v_i$ and $u_i u_{i+m/2}$ are in the right half plane spanned by the directed line from u_i to $u_{i+m/2}$, for each $1 \leq i \leq m$.

2.2 The construction of $N_{m,m,1}$

To construct the new drawing $N_{m,m,1}$ of K_{2m+1}, we start with the Harary-Hill drawing H_{2m} of K_{2m} and use the center w of the two concentric m-gons as additional vertex. We draw the edges from w to the other $2m$ vertices as straight line segments. Finally, we redraw all the edges of the form $u_i v_i$ such that they leave u_i to the outside of U, surround U (roughly half), and cross U at the edge $u_{\lfloor (m+1)/2 \rfloor} u_{\lfloor (m+3)/2 \rfloor}$. (If desired, we locally modify the drawing slightly so that no three edges cross at the same point.) Figures 3 and 4 show this construction for $m = 3$, 4, 5, and 6. Note that while

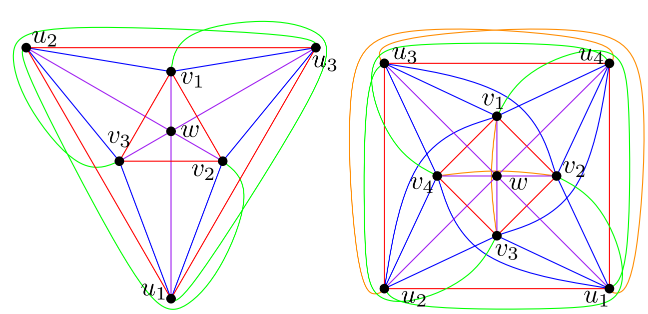

Figure 3: New family for $n = 7, 9$: $N_{3,3,1}$ and $N_{4,4,1}$.

for $m \leq 4$, the drawing $N_{m,m,1}$ contains edges that do not participate in any crossing, for $m \geq 5$, every edge is crossed by at least one other edge. This implies that each cell of the drawing $N_{m,m,1}, m \geq 5$, has at most one vertex on its boundary and thus the drawing is not s-shellable for any $s \geq 2$. For $m = 4$, the drawing $N_{4,4,1}$ is in fact 9-shellable (with the 9-shelling

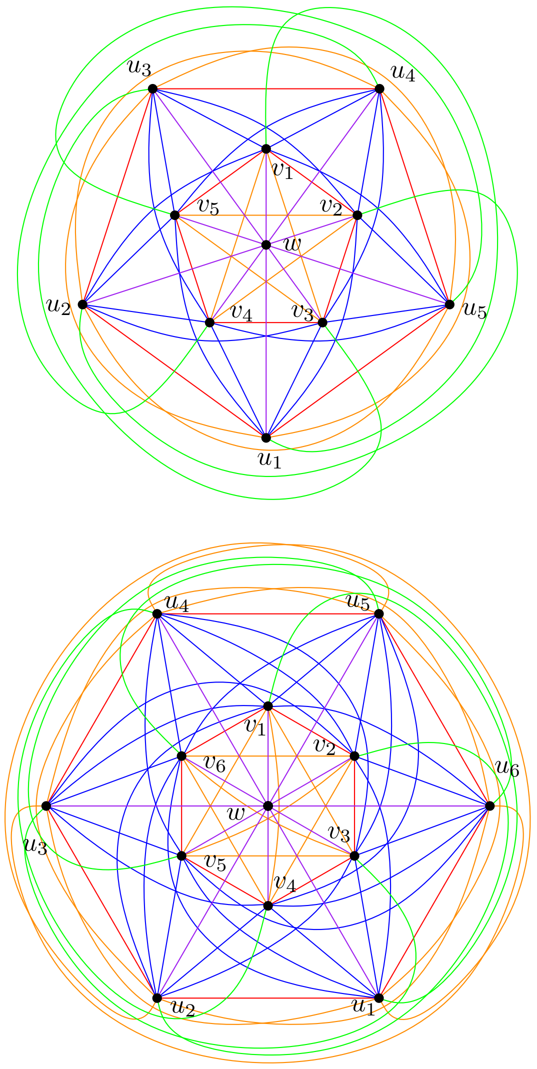

Figure 4: New family for $n = 11, 13$: $N_{5,5,1}$ and $N_{6,6,1}$.

$(w, v_2, v_1, u_4, u_3, v_4, u_2, u_1, v_3)$), and thus the members of our new family constitute non-shellable drawings for $n \geq 11$.

2.3 Counting the number of crossings in $N_{m,m,1}$

We start with $H(2m) = m(m-1)^2(m-2)/4$ crossings in H_{2m}. In both drawings, H_{2m} and $N_{m,m,1}$, we refer to the edges in $\{u_iv_i : 1 \leq i \leq m\}$ as green (they are also colored green in all figures). The crossings contribution of these re-routed edges is addressed last.

First, consider the *responsibility* of w in $N_{m,m,1}$, that is, the number of crossings in $N_{m,m,1}$ that involve the vertex w. Note that none of the edges incident to w crosses an edge u_ju_k, a green edge u_iv_i, or another edge incident to w.

We start by counting the number of crossings in which the path v_1wu_1 (formed by the edges v_1w and wu_1) is involved. The portion of v_1wu_1 contained between U and V is crossed exclusively by edges that connect U and V. More precisely, between U and V, wu_1 crosses exactly all edges $\{v_ju_k : 3 \leq j \leq \lfloor \frac{m+1}{2} \rfloor, 2 \leq k \leq j-1\}$ and $\{v_ju_k : \lfloor \frac{m+3}{2} \rfloor \leq j \leq m-1, j+1 \leq k \leq m\}$; see Figure 5. The total number of these crossings is

$$\sum_{j=3}^{\lfloor \frac{m+1}{2} \rfloor} (j-2) + \sum_{j=\lfloor \frac{m+3}{2} \rfloor}^{m-1} (m-j),$$

which is equal to

$$\frac{1}{2}\left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{m-3}{2} \right\rfloor + \frac{1}{2}\left\lceil \frac{m-1}{2} \right\rceil \left\lceil \frac{m-3}{2} \right\rceil.$$

The portion of v_1wu_1 inside (and on the boundary of) V is crossed exclusively by edges that connect two vertices of V. More precisely, these are all

$$\left\lfloor \frac{m-1}{2} \right\rfloor \left\lceil \frac{m-1}{2} \right\rceil$$

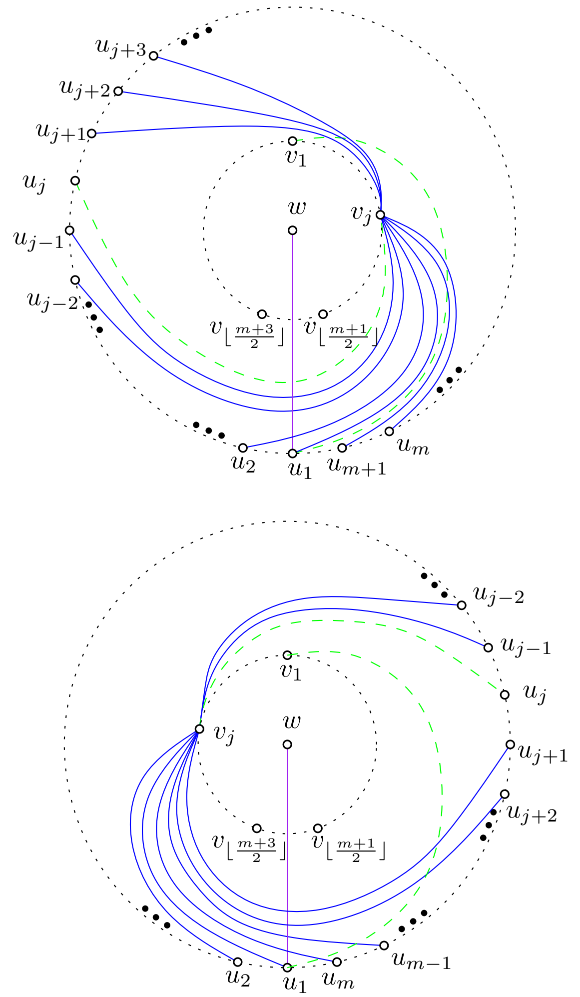

Figure 5: Counting crossings between edges wu_1 and u_iv_j: (top) For $3 \leq j \leq \lfloor \frac{m+1}{2} \rfloor$, v_ju_k crosses wu_1 if $2 \leq k \leq j-1$. (bottom) For $\lfloor \frac{m+3}{2} \rfloor \leq j \leq m-1$, v_ju_k crosses wu_1 if $j+1 \leq k \leq m$.

edges $\{v_j v_k : 2 \le j \le \lfloor \frac{m+1}{2} \rfloor, \lfloor \frac{m+3}{2} \rfloor \le k \le m\}$, plus, if m is even, the edge $v_1 v_{1+\frac{m}{2}}$.

Now consider the path $v_i w u_i$ for arbitrary i, $2 \le i \le m$. If m is odd, then by symmetry, the number of crossings in which $v_i w u_i$ is involved is the same as for $v_1 w u_1$. If m is even, then the number is the same for $2 \le i \le m/2$, and one less for $m/2 + 1 \le i \le m$ (as then $v_i w u_i$ is not crossed by $v_i v_{i-m/2}$). Hence the responsibility of w in $N_{m,m,1}$ is

$$\frac{m}{2}\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{m-3}{2}\right\rfloor+\frac{m}{2}\left\lceil\frac{m-1}{2}\right\rceil\left\lceil\frac{m-3}{2}\right\rceil + m\left\lfloor\frac{m-1}{2}\right\rfloor\left\lceil\frac{m-1}{2}\right\rceil \quad + \frac{m}{2} \text{ (if } m \text{ is even)},$$

which, by standard formula manipulation, can be rewritten as

$$m\binom{m-1}{2} \quad + \frac{m}{2} \text{ (if } m \text{ is even)}. \qquad (1)$$

Next we consider the crossings contribution of the green edges in H_{2m} and $N_{m,m,1}$, respectively. In H_{2m}, the edge $u_1 v_1$ crosses exactly all edges $\{v_j u_k : 2 \le j \le m-1, j+1 \le k \le m\}$; see again Figure 5. As none of the crossed edges is green and as, by symmetry, all green edges are crossed by the same number of edges, this gives a total of

$$m\sum_{j=2}^{m-1}(m-j) = m\binom{m-1}{2} \qquad (2)$$

crossings with green edges in H_{2m}.

For counting the number of crossings involving green edges in $N_{m,m,1}$, we use the fact that H_{2m} is "inside-out symmetric", in the sense that exchanging the roles of V and U such that the cell containing w becomes the unbounded cell and vice versa, again gives the same drawing. This implies that in $N_{m,m,1}$, the number of crossings of the green edge $u_1 v_1$ with non-green edges is the same as the number of crossings of the path $u_1 w v_1$ (with the only exception that for m even, the path $u_1 w v_1$ crosses $v_1 v_{m/2}$ while the edge $u_1 v_1$ does not cross the according edge $u_1 u_{m/2}$). By symmetry of $N_{m,m,1}$, all green edges of $N_{m,m,1}$ cross the same number of non-green edges. Further, when m is odd, any two green edges cross in $N_{m,m,1}$. If m is even, all but $\frac{m}{2}$ pairs of green edges cross in $N_{m,m,1}$, namely, $v_i u_i$ does not cross $v_{i+m/2} u_{i+m/2}$. Hence, the total crossings contribution of the green edges in $N_{m,m,1}$ is

$$m\binom{m-1}{2}+\binom{m}{2}-\frac{m}{2} \text{ (if } m \text{ is even)}. \qquad (3)$$

Finally, we can combine the number $H(2m)$ of crossings in H_{2m} with the results from Equations (1), (2), and (3): Altogether, regardless of the parity of m, there is a total of

$$H(2m)+\left[m\binom{m-1}{2}+\frac{1}{4}(1+(-1)^m)m\right]-m\binom{m-1}{2} + \left[m\binom{m-1}{2}+\binom{m}{2}-\frac{1}{4}(1+(-1)^m)m\right] = \frac{1}{4}m^2(m-1)^2 = H(2m+1)$$

crossings in $N_{m,m,1}$.

2.4 The drawings $N_{m,m,2}$ and $N_{m,m,3}$

For m odd, we now generate the drawing $N_{m,m,2}$ of K_{2m+2} with exactly $H(2m+2)$ crossings, and the drawing $N_{m,m,3}$ of K_{2m+3} with exactly $H(2m+3)$ crossings.

We start from $N_{m,m,1}$ and duplicate the vertex w to obtain $N_{m,m,2}$; cf. Figure 6. We denote the new vertex by w_1. This can be done in such a way that there are exactly $m(m-1)$ crossings involving only edges incident to w or w_1 in $N_{m,m,2}$, the edge ww_1 is not intersected by any edge, and deleting w from $N_{m,m,2}$ gives a copy of $N_{m,m,1}$. Then the responsibility of w_1 in $N_{m,m,2}$ is the responsibility of w in $N_{m,m,1}$ plus $m(m-1)$. That is, $N_{m,m,2}$ has exactly $H(2m+1)+m\binom{m-1}{2}+m(m-1) = m^2(m^2-1)/4 = H(2m+2)$ crossings.

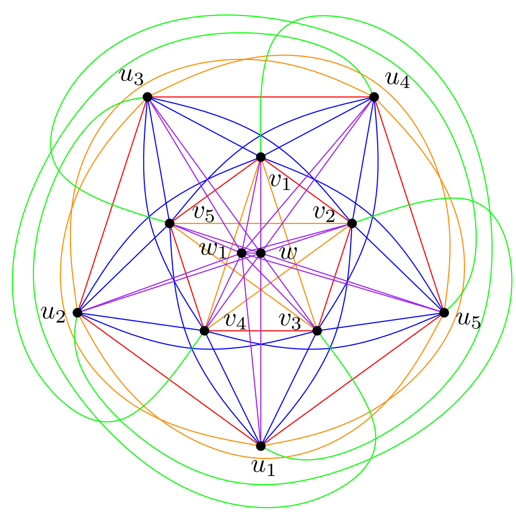

Figure 6: Drawing of $N_{5,5,2}$.

Further, we duplicate w in $N_{m,m,2}$ to obtain $N_{m,m,3}$; see Figure 7. We call this new vertex w_2. This can be done so that there are exactly m^2 crossings involving only edges incident to w or w_2 in $N_{m,m,3}$, the edge ww_2 is not intersected by any edge, and deleting w or w_2 from $N_{m,m,3}$ gives a copy of $N_{m,m,2}$. Then the responsibility of w_2 in $N_{m,m,3}$ is the responsibility of w in $N_{m,m,2}$ plus m^2. That is, $N_{m,m,3}$ has exactly $H(2m+2)+m\binom{m-1}{2}+m(m-1)+m^2 = m^2(m+1)^2/4 = H(2m+3)$ crossings.

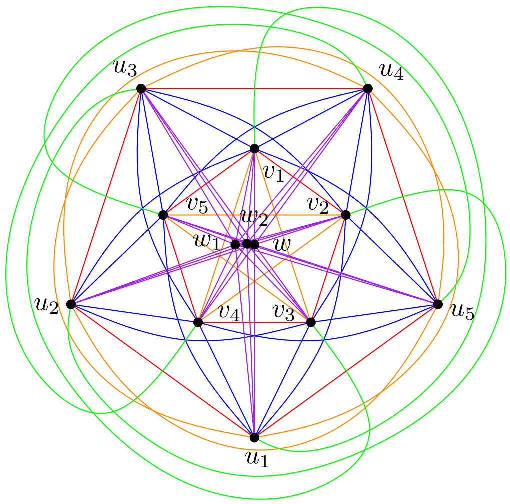

Figure 7: Drawing of $N_{5,5,3}$.

2.5 Some observations about the new family

The drawings $N_{m,m,1}$ have an odd number of vertices. Note that for $m \geq 5$ every edge of $N_{m,m,1}$ is intersected at least once: In the corresponding H_{2m}, all non-green edges except for the edges of the polygons U and V are crossed by at least one non-green edge. Hence, they remain crossed in $N_{m,m,1}$. Further, in $N_{m,m,1}$, each edge of V is crossed by an edge wu_i, and each edge of U is crossed by a green edge. Because all cells determined by the drawing $N_{m,m,1}$ contain at most one vertex, the drawing is non-shellable.

The number of vertices in the drawings $N_{m,m,2}$, for m odd, is a multiple of 4. In these drawings, the edge ww_1 is not crossed by other edges, but for $m \geq 5$ all remaining edges are crossed. We do not have a new drawing of K_n with $n \equiv 2 \mod 4$. The drawing $N_{m,m,3}$, is different from $N_{m+1,m+1,1}$ since the edge ww_2 is not crossed in $N_{m,m,3}$ and all edges are crossed in $N_{m+1,m+1,1}$. Finally, due to the construction and the symmetry of the drawings, it is not hard to see that for $m \geq 5$, none of $N_{m,m,2}$ and $N_{m,m,3}$ is shellable. (They are not s-shellable for $s \geq 3$. They are 2-shellable with 2-shelling w, w_1 in $N_{m,m,2}$ and w, w_2 in $N_{m,m,3}$.)

3 Crossing optimal drawings for small graphs

Considering sets of small, constant cardinality can help to see generic patterns behind their structure, which might generalize to arbitrary n. We will show below that our new family from Section 2 already shows up in examples for 7 and 8 vertices and, together with the known families, covers all essentially different crossing optimal drawings with cardinality at most 8.

Good drawings of K_n can be classified into isomorphism classes with similar combinatorial properties. More precisely, two good drawings are isomorphic if they are homeomorphic on the sphere, and weakly isomorphic if the same set of pairs of edges properly intersect [5]. It is well known that these crossing properties are covered by the rotation system of K_n (see for example [5], Proposition 6): the rotation system of a drawing of K_n gives for each vertex v of K_n the clockwise circular ordering around v of all edges incident to v.

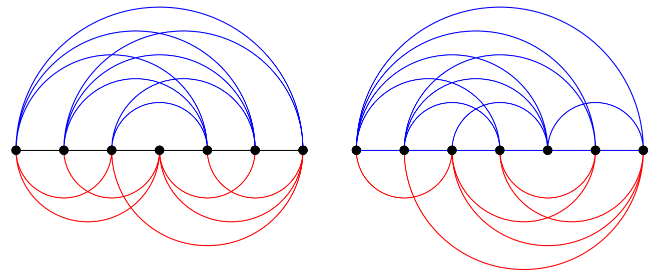

Figure 8: Two of the three crossing optimal 2-page book drawings with $n = 7$ vertices.

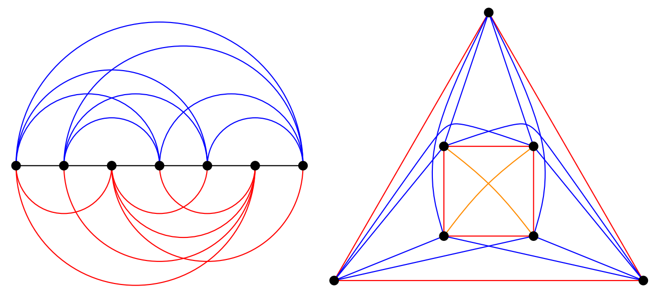

Figure 9: Two different drawings of the same rotation system for $n = 7$ vertices (a 2-page book drawing and a cylindrical drawing).

Since we are essentially interested in the crossing properties of good drawings of K_n it is therefore sufficient to consider only the different (weak isomorphism) classes, that is, rotation systems, instead of all different possible drawings. As the order in which an edge is intersected by some other edges is not relevant for weakly isomorphic drawings, one rotation system might be realized by (exponentially) many different drawings. But all of them have the same (number of) crossings.

It is instructive to have a look at the different possibilities for small graphs. For any $n \leq 6$, there is only one crossing optimal rotation system for K_n, and it can be drawn as 2-page book drawing; see Figure 2 for $n = 5, 6$.

For $n = 7$ there are 5 different crossing optimal rotation systems. Three of them can be drawn as 2-page book drawings, where one can at the same time be drawn as a cylindrical drawing; see Figures 8 and 9. The drawings of the two remaining rotation systems are related to our new construction. See Figure 3 (left) for $N_{3,3,1}$ and Figure 10 for a variation where the edges u_iv_i of the related Harary-Hill drawing of K_6 have not been modified. We remark that the latter drawing can be generalized for any $n = 2m + 1 \geq 7$: add the center w to the Harary-Hill drawing of K_{2m} and connect it to the $2m$ other points with straight lines. Following the arguments of Section 2.3 shows that this gives a family of drawings with $H(2m+1)$ crossings. However, as the cycle $u_1, u_2, \ldots, u_n$ consists of non-crossed edges, these drawings are $\lfloor \frac{n}{2} \rfloor$-shellable, and thus we do not go into further detail.

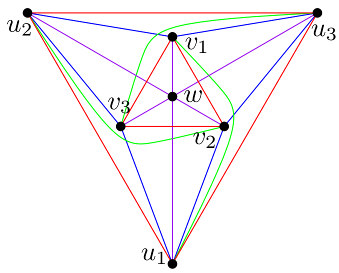

Figure 10: A drawing of one of the two non-cylindrical non 2-page book rotation systems for $n = 7$ vertices.

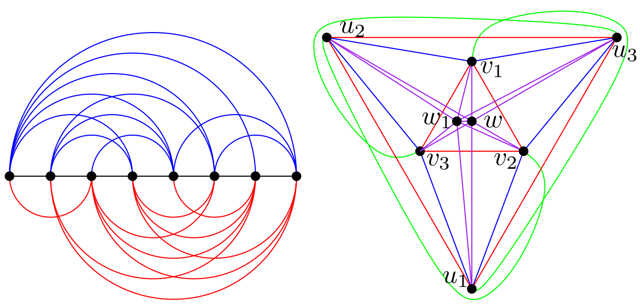

Figure 11: Crossing optimal drawings with $n = 8$ vertices: (left) 2-page book drawing, (right) $N_{3,3,2}$.

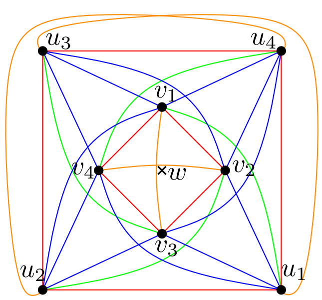

Figure 12: Crossing optimal cylindrical drawing with $n = 8$ vertices.

For $n = 8$, there are three different crossing optimal rotation systems. One can be drawn as a 2-page book embedding, one as a cylindrical drawing, and the third rotation system can be drawn as $N_{3,3,2}$. See Figures 11 and 12.

As mentioned before, for $m = 5$, the construction of our new family results in a crossing optimal drawing (and thus rotation system) for K_{11} that does not contain any non-crossed edge (Figure 4 (top)). There are three other crossing optimal rotation systems for $n = 11$ with this property – out of a total of 403079 crossing optimal rotation systems. For smaller cardinality, no such example exists.

4 Conclusion

In this paper we presented several new classes of drawings of the complete graph K_n with the conjectured minimum number $H(n)$ of crossings. On the one hand, our drawings constitute a new infinite family of graphs with this property, thus extending the two existing families of 2-page book drawings and Harary-Hill cylindrical drawings. On the other hand, and maybe most importantly, our family is the first infinite class of graphs with $H(n)$ crossings which is not s-shellable for any $s \geq 2$. It thus sheds new light on (possibly crossing optimal) constructions beyond the shellable ones. The hope is that this start of a systematic characterization helps to extend the lower bound arguments of [1] to a broader class of drawings, with the ultimate goal to prove the Harary-Hill Conjecture in its full generalization.

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