There is a unique crossing-minimal rectilinear drawing of K_{18}

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Abstract

We show that, up to isomorphism, there is a unique crossing-minimal rectilinear drawing of K_{18}. As a consequence we settle, in the negative, the following question from Aichholzer and Krasser: does there always exist an crossing-minimal drawing of K_n that contains a crossing-minimal drawing of K_{n-1}?

Keywords: Rectilinear crossing number, complete graphs, k-edges

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1 Introduction

The *rectilinear crossing number* $\overline{\text{cr}}(G)$ of a graph G is the minimum number of edge crossings in a *rectilinear* (or *geometric*) drawing of G in the plane, i.e., a drawing of G in the plane where the vertices are points in general position and the edges are straight segments. A drawing of G with exactly $\overline{\text{cr}}(G)$ crossings is *crossing-minimal*.

Determining the rectilinear crossing number $\overline{\text{cr}}(K_n)$ of the complete graph K_n is a well-known open problem in combinatorial geometry (see for instance [9]). In [7] Aichholzer et al. announced the exact determination of $\overline{\text{cr}}(K_n)$ for $13 \leq n \leq 17$. In that paper also the following question was raised.

Question 1.1 *Is it true that, for every integer $n \geq 4$, there exists an crossing-minimal drawing of K_n that contains an crossing-minimal drawing of K_{n-1}?*

The exact value of $\overline{\text{cr}}(K_n)$ is known for $n \leq 27$ and $n = 30$ (see [1,5,6,7,8]). In particular, $\overline{\text{cr}}(K_{18}) = 1029$ was established in [6]. Crossing-minimal rectilinear drawings of K_n for this range of values of n can be found in [2] and [4].

Let θ denote the counterclockwise rotation of $2\pi/3$ around the origin, and let $W := \{(-51, 113), (6, 834), (16, 989), (18, 644), (18, 1068), (22, 211)\}$. Then (see [2]) the 18–point set $W \cup \theta(W) \cup \theta^2(W)$ induces an crossing-minimal drawing of K_{18}.

Our main result is the following.

Theorem 1.2 *Up to order type isomorphism, there is a unique 18-point set whose induced rectilinear drawing of K_{18} has $\overline{\text{cr}}(K_{18})$ crossings.*

Let $\mathcal{D}$ be the (unique, in view of Theorem 1.2) crossing-minimal geometric drawing of K_{18}. It is easily verified that every subdrawing of $\mathcal{D}$ with 17 points has more than $\overline{\text{cr}}(K_{17}) = 798$ crossings. This settles Question 1.1 in the negative.

In the next section, we introduce the necessary notation and additional concepts required for the proof of Theorem 1.2. In Section 3 we give a brief sketch of the proof of Theorem 1.2.

2 k-edges, $(\leq k)$-edges, and 3-decomposability

Let Q be a point set in the plane. If $p, q \in Q$, we denote by pq the straight line segment with end points p and q. We use $\ell(pq)$ to denote the directed line that spans p and q, directed from p towards q. Furthermore, $\ell(pq)^+$ and $\ell(pq)^-$ denote the halfplanes to the right and left, respectively, of $\ell(pq)$.

Let Q be an n-point set in the plane in general position, and let $0 \leq k \leq n/2-1$. A *k-edge* of Q is a line that spans two points of Q, and leaves exactly k points on one side. A $(\leq k)$-edge (respectively, a $(> k)$-edge) is an i-edge with $0 \leq i \leq k$ (respectively, $k < i \leq n/2 - 1$). Let $E_k(Q), E_{\leq k}(Q)$, and $E_{>k}(Q)$ denote, respectively, the number of k-edges, $(\leq k)$-edges and $(> k)$-edges of Q. Note that $E_{\leq k}(Q) = \sum_{j=0}^{k} E_j(Q)$ and $E_{>k}(Q) = \binom{n}{2} - E_{\leq k}(Q)$. The vector $v_k(Q) := (E_0(Q), E_1(Q), \ldots, E_{\lfloor n/2 \rfloor -1}(Q))$ is the *vector of k-edges* of Q. The *vector $v_{\leq k}(Q)$ of $(\leq k)$-edges* of Q is analogously defined. Finally, $E_{\leq k}(n)$ denotes the minimum of $E_{\leq k}(Q)$ taken over all n-point sets Q. The exact determination of $E_{\leq k}(n)$ is another open problem in combinatorial geometry (see [1,3,5,6]).

The number of crossings in a geometric drawing of K_n and the number of k- and $(\leq k)$-edges in its underlying n-point set P are closely related by the following equality, independently proved in [3] and [10]:

$$\overline{\mathrm{cr}}(P) = \sum_{k=0}^{\lfloor n/2 \rfloor -1} (n-2k-3)\, E_{\leq k}(P) - \frac{3}{4}\binom{n}{3} + \left(1 + (-1)^{n+1}\right) \frac{1}{8}\binom{n}{2}. \quad (1)$$

Finally, we introduce a concept that captures a property shared by all known crossing-minimal geometric drawings of K_n, for n a multiple of 3. A point set P is 3-*decomposable* if it can be partitioned into three equal-size sets A, B and C, such that (i) there exist a triangle T enclosing the point set P; and (ii) the orthogonal projection of P onto the three sides of T shows A between B and C on one side, B between C and A on the second side, and C between A and B on the third side. In this context, $\{A, B, C\}$ is a 3-*decomposition* of P.

3 Uniqueness of crossing-minimal drawing of K_{18}

Throughout this section, $\mathcal{D}$ is an crossing-minimal rectilinear drawing of K_{18}, and P is its underlying 18-point set.

Our strategy is as follows. First we show that the crossing-minimality of $\mathcal{D}$ completely determines $v_{\leq k}(P)$. We then argue that the entries of $v_{\leq k}(P)$ imply that P must be 3-decomposable. This in turn allows us to classify certain (as it happens, many) types of k-edges that must occur in P. Finally, we find a set of restrictions on the remaining k-edges, and show that they uniquely determine the order type of P.

Sketch of Proof of Theorem 1.2 Using (1) and $\overline{\text{cr}}(K_{18}) = 1029$, it is not difficult to prove the following.

Proposition 3.1 $v_{\leq k}(P) = (3, 9, 18, 30, 45, 63, 87, 120, 153)$.

We start by labeling the points of P. Since $E_0(P) = 3$, then the convex hull of P consists of exactly 3 points, say a_6, b_6 and c_6. Now we rotate $\ell(a_6c_6)$ from c_6 to b_6 around a_6 and for $i = 1, 2, \ldots, 5$, we let c_{6-i} be the i-th point found by such a rotation. Similarly, we rotate $\ell(a_6b_6)$ from b_6 to c_6, again around a_6, and for $i = 1, 2, \ldots, 5$ we let b_{6-i} be the i-th point found by such a rotation. Let $C := \{c_1, \ldots, c_6\}$, $B := \{b_1, \ldots, b_6\}$ and $A := P \setminus B \cup C$. Clearly, $\{A, B, C\}$ is a partition of P.

From the entries of $v_{\leq k}(P)$ it follows that the same partition of P is obtained if, instead of rotating around a_6, we rotate around b_6 or c_6. Moreover, for $\{x, y, z\} = \{a, b, c\}$, the numbers in $v_{\leq k}(P)$ imply that the rotations of $\ell(y_6x_6)$ and $\ell(z_6x_6)$ around y_6 and z_6, respectively, produce the same labels for the x's points, and so this labeling is well-defined. Note that $\{A, B, C\}$ is a 3-decomposition of P. In this context we define, as in [2], two types of edges. Let $p, q \in P$. If $p, q \in A$, $p, q \in B$ or $p, q \in C$ then we call pq *monochromatic*; otherwise, pq is *bichromatic*. Let $E^{mono}_{\leq k}(P)$ and $E^{bi}_{\leq k}(P)$ be the number of monochromatic and bichromatic $(\leq k)$-edges of P, respectively. Note that $E_k(P) = E^{mono}_k(P) + E^{bi}_k(P)$.

For $x \in \{a, b, c\}$, let us denote the number of monochromatic $(> k)$-edges of type xx by $E^{xx}_{>k}(P)$. Note that $E^{mono}_{>k}(P) = E^{aa}_{>k}(P) + E^{bb}_{>k}(P) + E^{cc}_{>k}(P)$.

Remark 3.2 *Let $\{x, y, z\} = \{a, b, c\}$. Clearly, if we rotate $\ell(x_6y_6)$ around x_6 from y_6 to z_6, and $x_{\sigma(i)}$ is the i-th x that is found by such a rotation, then $x_6x_{\sigma(i)}$ is a j-edge of P for $j = min\{5 + i, 16 - (5 + i)\}$. Thus for $j = 6, 7$ there are exactly two j-edges of the type x_6x, and for $j = 8$ there is exactly one 8-edge of the type x_6x.*

The following is an immediate consequence of the 3-decomposability of P, Claim 1 in [2], and the fact that $E^{bi}_8(P) = 108 - \sum_{i=0}^{7} E^{bi}_i(P)$.

Proposition 3.3 $E^{bi}_k(P) = 3(k + 1)$ *for* $k = 0, \ldots, 5$*;* $E^{bi}_k(P) = 18$ *for* $k = 6, 7$*; and* $E^{bi}_8(P) = 9$.

Propositions 3.1 and 3.3 readily imply the following.

Corollary 3.4 $E^{mono}_8(P) = 24$*,* $E^{mono}_7(P) = 15$*,* $E^{mono}_6(P) = 6$*, and* $E^{mono}_k(P) = 0$ *for* $k = 0, 1, 2, 3, 4,$ *and* 5.

Claim 4 in [2] implies that $E^{xx}_8(P) \leq 8$ for each $x \in \{a, b, c\}$. Using this, together with Remark 3.2 and Proposition 3.4, we obtain the following.

Proposition 3.5 *Let $x \in \{a, b, c\}$. Then $E_6^{xx}(P) = 2$, $E_7^{xx}(P) = 5$, and $E_8^{xx}(P) = 8$. Moreover, each 6-edge of type xx involves x_6.*

Using this last result and similar arguments, we obtain the following.

Proposition 3.6 *Let $x \in \{a, b, c\}$. Then 1) x_6x_5 cannot be a 6-edge; 2) there are at least two 7-edges of type xx involving x_5 but not x_6; 3) x_6x_4 cannot be a 6-edge; 4) each element of $\{x_3x_2,\ x_3x_1, x_2x_1\}$ is an 8-edge; and 5) x_6x_2 and x_6x_1 are the two 6-edges of type xx, i.e., x_3, x_4 and x_5 are contained in the triangle formed by x_6, x_2 and x_1.*

We may assume that $x_2 \in \ell(x_6x_1)^-$. Thus the triangle formed by x_6, x_2 and x_1 is as in Figure 1. Since $\ell(x_3, x_1)$ is an 8-edge, then exactly one element, say w, of $\{x_4, x_5, y_1, \ldots, y_5\}$ belongs to $\ell(x_3, x_1)^-$. A tedious but straightforward case analysis shows that w must be y_1. Similarly, since $\ell(x_2, x_1)$ is an 8-edge, then there are exactly two y's in $\ell(x_2, x_1)^-$. Clearly, one of them is y_1. We can then deduce that the other y must be y_2 and that $y_2 \in \ell(y_6y_1)^-$. Then the points of P with indices $1, 2, 3$ and 6 are as in Figure 1.

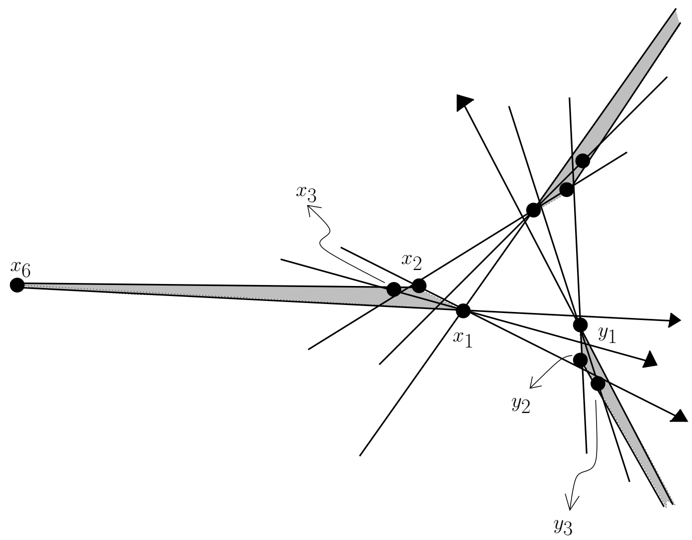

Fig. 1. The relative position of the points of P with indices $1, 2, 3$ and 6.

Using similar arguments one can prove the following.

Proposition 3.7 *Let $x \in \{a, b, c\}$. Then 1) x_4x_2 cannot be an 8-edge; 2) x_6x_5 is an 8-edge; 3) $x_3 \in \ell(x_6x_5)^-$ and $x_4 \in \ell(x_6x_5)^+$; 4) $x_4 \in \ell(x_2x_3)^-$ and $x_5 \in \ell(x_2x_3)^+$; and 5) $x_5 \in \ell(x_1x_4)^-$ and $x_3 \in \ell(x_1x_4)^+$.*

Finally, it is not difficult to see that the order type of P is uniquely determined by the labeling of the points of P, the numbers in $v_{\leq k}(P)$, and the set of restrictions given by Propositions 3.5, 3.6, and 3.7. □

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