Improved lower bounds on book crossing numbers of K_n

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Abstract

A *k-page book drawing* of a graph G is a drawing of G on k halfplanes in the space with common boundary l, a line, where the vertices are on l and the edges cannot cross l. The k-page book crossing number of G, denoted by $\nu_k(G)$, is the minimum number of edge-crossings over all k-page book drawings of G. We improve the lower bounds on $\nu_k(G)$ for all $k \geq 15$ and determine $\nu_k(G)$ whenever $2 < n/k \leq 3$. Our proofs rely on bounding the number of edges in convex graphs with small local crossing numbers.

1 Introduction

In a *k-page book drawing* of a graph, the vertices are placed on a line l and each edge is completely contained in one of k fixed halfplanes in the space whose boundary is l. The *k-page book crossing number* of a graph G, denoted by $\nu_k(G)$, is the minimum number of edge-crossings over all k-page book drawings of G. Book crossing numbers have been studied in relation to their applications in VLSI designs. We are concerned with the k-page book crossing number of the complete graph K_n. In 1964, Blažek and Koman [2] described k-page book drawings of K_n with few crossings. They described their construction in detail only for $k = 2$, explicitly computed its crossing number for $k = 2$ and 3, and implicitly conjectured that generalizations of these constructions to larger values of k achieved $\nu_k(K_n)$. In 1994, Damiani et al. [3] described constructions using adjacency matrices, and in 1996, Shahrokhi et al. [5] provided a geometric description of similar k-page book drawings of K_n. In 2013, de Klerk et al. [4] gave another construction whose number of crossings is

$$Z_k(n) := r \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor + 1, n\right) + (k - r) \cdot F\left(\left\lfloor \frac{n}{k} \right\rfloor, n\right)$$

where $F(q, n) = q(q^2 - 3q + 2)(2n - 3 - q)/24$ and $r = (n \bmod k)$. Then

$$\nu_k(K_n) \leq Z_k(n) = \left(\frac{2}{k^2}\left(1 - \frac{1}{2k}\right)\right)\binom{n}{4} + O(n^3).$$

All the constructions in [3], [5], and [4] generalize the original Blažek-Koman construction but are slightly different. They are widely believed to be asymptotically correct giving rise to the following conjecture.

Conjecture 1. *For any positive integers k and n, $\nu_k(K_n) = Z_k(n)$.*

Ábrego et al. [1] proved this conjecture for $k = 2$. The only other previously known values of $\nu_k(K_n)$ are $\nu_k(K_n) = 0$ for $k > \lceil n/2 \rceil$ and a few sporadic values for $n \leq 15$ and $k \leq 5$ [4]. We prove the conjecture for any k and n such that $2 < n/k \leq 3$ (Theorem 5), and give improved lower bounds for $n/k > 3$ (Theorem 4). Shahrokhi et al. [5] proved the lower bound

$$\nu_k(n) \geq \frac{n(n-1)^3}{296k^2} - \frac{27kn}{37} = \frac{3}{37k^2}\binom{n}{4} + O(n^3),$$

which was later improved by de Klerk et al. [4] to

$$\nu_k(K_n) \geq \begin{cases} \frac{3}{119}\binom{n}{4} + O(n^3) \text{ if } k = 4, \\ \frac{2}{(3k-2)^2}\binom{n}{4} \text{if } k \text{ even}, n \geq \frac{k^2}{2} + 3k - 1, \\ \frac{2}{(3k+1)^2}\binom{n}{4} \text{ if } k \text{ odd}, n \geq k^2 + 2k - \frac{7}{2}. \end{cases} \quad (1)$$

Using semidefinite programming, they further improved the lower bound for several values of $k \leq 20$. We improve their bounds for $15 \leq n \leq 20$ as well as the asymptotic bound (1) for every k (Theorem 6).

2 Maximum number of edges

Our results heavily rely on a different problem on convex graphs. Let G_n be the rectilinear drawing of K_n whose vertices are the vertices of the regular n-gon. A *convex graph* can be defined as any subdrawing of G_n. To study crossings, it is convenient to disregard the sides of the polygon as edges. Let

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D_n be obtained form G_n by removing the sides of the polygon. Let $e_\ell(n)$ be the maximum number of edges over all convex subgraphs of D_n such that each edge is crossed at most ℓ times. Brass et al. studied the problem of maximizing the number of edges over convex graphs satisfying certain crossing conditions. Functions equivalent to $e_\ell(n)$ for general drawings of graphs in the plane were studied by Ackerman, and Pach et al. We determined the exact values of $e_\ell(n)$ for $\ell \le 3$ and any n.

Theorem 2. *For any $n \ge 3$,*

$$\begin{aligned} e_0(n) &= n-3, \\ e_1(n) &= \tfrac{3}{2}(n-3)+\delta_1(n), \\ e_2(n) &= 2(n-3)+\delta_2(n), \\ e_3(n) &= \tfrac{9}{4}(n-3)+\delta_3(n), \end{aligned}$$

where

$$\delta_1(n) = \begin{cases} 1/2 & \text{if } 2|n, \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_2(n) = \begin{cases} 1 & \text{if } 3|(n-2), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\delta_3(n) = \begin{cases} -1/4 & \text{if } n \equiv 0 \pmod 4, \\ 1/2 & \text{if } n \equiv 1 \pmod 4, \\ 5/4 & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

3 Crossings in k-page books

For any integers $k \ge 1$, $n \ge 3$, and $m \ge 0$, define $L_{k,n}(m) = \frac{m}{2}n(n-3) - k\sum_{\ell=0}^{m-1} e_\ell(n)$.

Theorem 3. *Let $n \ge 3$ and $k \ge 3$ be fixed integers. Then, for all integers $m \ge 0$, $\nu_k(K_n) \ge L_{k,n}(m)$. The value of $L_{k,n}(m)$ is maximized by the smallest m such that $e_m(n) \ge \frac{n(n-3)}{2k}$.*

We explicitly state the best bounds guaranteed by Theorem 3 and the values of $e_\ell(n)$ in Theorem 2.

Theorem 4. *For any $k \ge 3$ and $n \ge 2k$,*

$$\nu_k(K_n) \ge \begin{cases} \frac{1}{2}(n-3)(n-2k) \text{ if } 2k < n \le 3k, \\ (n-3)(n-\frac{5}{2}k) - \delta_1(n)k \\ \qquad\qquad \text{if } 3k < n \le 4k, \\ \frac{3}{2}(n-3)(n-3k) - (\delta_1+\delta_2)(n)k \\ \qquad \text{if } 4k < n \le \begin{cases} \lceil 4.5k \rceil - 1 \text{ if } 4|n, \\ \lfloor 4.5k \rfloor \text{ otherwise,} \end{cases} \\ 2(n-3)(n-\frac{27}{8}k) - (\delta_1+\delta_2+\delta_3)(n)k \\ \qquad\qquad \text{otherwise.} \end{cases}$$

The first part of Theorem 4 settles Conjecture 1 when $2 < \frac{n}{k} \le 3$.

Theorem 5. *If $2 < \frac{n}{k} \le 3$, then $\nu_k(K_n) = \frac{1}{2}(n-3)(n-2k)$.*

The bound in Theorem 4 becomes weaker as n/k grows. We use a different approach to improve this bound when n is large with respect to k. Using Theorem 3, for fixed k and for all $n \ge n' \ge 4$,

$$\frac{\nu_k(K_n)}{\binom{n}{4}} \ge \frac{\nu_k(K_{n'})}{\binom{n'}{4}} \ge \max_{\substack{1\le m\le 4 \\ n' \ge 2k}} \frac{L_{k,n'}(m)}{\binom{n'}{4}}.$$

We use $n' = \lfloor \frac{81}{16}k \rfloor$, which optimizes the previous lower bound when $k \equiv 3$, 11, 15, 18, 22, 30, 37, 41, 48, 56, 60 (mod 64) and is close to optimal for all other values of k. The universal bound given in Theorem 6 is obtained when $k \equiv 29$ (mod 64) and it is the minimum of the maxima over all classes mod 64. This result improves the asymptotic bound (1) for every k. In fact, the ratio of the lower to the upper bound on $\lim_{n\to\infty} \frac{\nu_k(K_n)}{\binom{n}{4}}$ is improved from approximately $\frac{1}{9} \approx 0.1111$ to $\frac{2024}{81^2} \approx 0.3089$.

Theorem 6. *For any $k \ge 3$ and $n \ge \lfloor 81k/16 \rfloor$, $\nu_k(n) \ge \left(\left(\frac{8}{9}\right)^4 \frac{1}{k^2} + \left(\frac{2}{3}\right)^{15} \frac{118}{k^3} + \Theta\left(\frac{1}{k^4}\right)\right)\binom{n}{4}$.*

Finally, using $n' = \lfloor \frac{81}{16}k \rfloor$, we improved the bounds in [4] (Table 4.3) for $15 \le k \le 20$.

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