New bounds on the maximum number of locally non-overlapping triangles in the plane.

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Abstract

We present new bounds on a problem by P. Brass on the maximum number of triangles that a drawing of a 3-uniform hypergraph in the plane can have so that any two triangles incident to a vertex do not have any other point in common.

1 Introduction

In his book, jointly co-authored with W. Mosser and J. Pach, P. Brass presented the following extremal geometric graph theory problem [3]. Related problems and results can be found in [1, 2].

Section 9.8, Problem 4 (Brass) What is the maximum number of hyperedges in a two-dimensional geometric three-hypergraph with n vertices in which no two edges incident to a vertex have any other point in common?

For concreteness, we denote that number as $f(n)$, and we denote the required property as the *local non-overlapping* property. It is observed that by choosing half of the interior faces of a maximal triangulation, one gets a set of triangles satisfying the local non-overlapping property. Thus, $f(n) \geq n-2$. By adding the interior angles of the triangles and observing that two triangles incident to a vertex must leave some uncovered angle between them, the upper bound $f(n) \leq 2n - O(1)$ is easily obtained.

In this note we narrow the gap between the two bounds by giving a better lower bound: $f(n) \geq 2n - c \cdot \log^3(n)$. We do this by first restricting the problem to the case when the set of n points is in convex position. We denote as $f^{conv}(n)$ the corresponding number. We give a construction that shows that $f^{conv}(n) \geq n - c \cdot \log_2^2(n)$. We then show how to adapt this convex-case construction to one for the general case, yielding our claimed bound.

We also list the exact values for $f^{conv}(n)$ for n up to 35.

2 Points in convex position

The problem of determining $f^{conv}(n)$, the maximum number of triangles that can be drawn with vertices on a set of n points in convex position, with the local non-overlapping property, turns out to be very interesting by itself. It is clear that any such set of triangles can be redrawn on the vertices of a regular n-gon, maintaining its local non-overlapping property. We thus might consider w.l.o.g. that the supporting point set is the set of vertices of a regular n-gon. The next easy lemma turns to be technically very helpful to tackle the problem. We denote the triangle with vertices $p, q,$ and r as $\Delta(p, q, r)$.

Lemma 1. $f^{conv}(n) \leq f^{conv}(n+1) \leq f^{conv}(n)+1$.

Proof. The left inequality is clearly true. We prove the second inequality.

Let T be a set of locally non-overlapping triangles whose vertices are those of a regular $(n+1)$-gon with circumcircle C. Let α be the maximum interior angle of any triangle in T. Let $t = \Delta(p, q, r)$ be one triangle s.t. $\angle rpq$ has size α. We claim that t is the only triangle incident to p. Suppose on the contrary that there exists another triangle $t' = \Delta(p, s, t)$ incident to p. Suppose w.l.o.g. that the points p, s, t, q, r appear in this order along the convex hull of the point set. Since the arc of C oposite to $\angle rpq$ is contained in the arc of C oposite to $\angle pst$, then $\angle pst$ is larger than $\angle rpq$, which has the maximum value α, a contradiction.

Now consider a set of $f^{conv}(n+1)$ triangles on $n+1$ points. By the previous observation, there exists one point p with only one triangle t incident to it. Remove both p and t to get a set of $f(n+1)-1$ triangles on n points. Thus, $f(n) \geq f(n+1)-1$. □

Theorem 1. $f^{conv}(n) \geq n - c \cdot \log^2(n)$.

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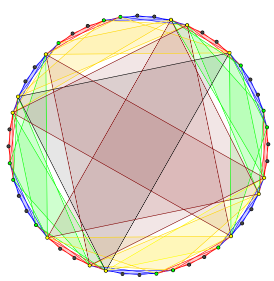
Proof (sketch). Let k be a positive integer and $n = 3(k+1)2^k$. We iteratively construct a set of n points with at least $n-(k+2)(k+1)$ locally non-overlapping triangles. This shows that $f(n) \geq n-(k+2)(k+1) > n-\log_2^2(n)$.

For stage t, we start with a set P_t of n_t points in convex position, where P_1 is the original set of points and $n_1 = n$. Write $n_t = 2t \cdot q_t + r_t$, where q_t and r_t are integers, and $0 \leq r_t < 2t$. Label the points in clockwise order $\{0, 1, 2, \ldots, n_t - 1\}$. For integers i and j, draw the triangle $T_{i,j} = \triangle(2t \cdot i + j, 2t \cdot i + j + t, 2t \cdot i + j + 2t)$, whenever $0 \leq i \leq q_t - 1$ and $0 \leq j \leq t-1$. As long as n_t is even, let M_t be the set of points of the form $2t \cdot i + j + t$ where $0 \leq i \leq q_t - 1$ and $0 \leq j \leq t-1$, or $2t \cdot q_t + j$ with $0 \leq j < r_t/2$. Let $P_{t+1} = P_t - M_t$. Then $n_{t+1} = |P_{t+1}| = n_t - tq_t - r_t/2 = 2t \cdot q_t + r_t - tq_t - r_t/2 = tq_t + r_t/2 = n_t/2$ and thus $n_{t+1} = n/2^t$. Also, stage t adds $tq_t = n_t/2 - r_t/2 = n/2^t - r_t/2 > n/2 - 2t$ triangles. Finally, relabel the points of P_{t+1} from 0 to $n_{t+1} - 1$, clockwise, in such a way that vertex 0 keeps its label throughout the process. Repeat this process until obtaining P_{k+1}, which has $n_{k+1} = n/2^k = 3(k+1)$ points labeled 0 to $3k+2$. Finally, draw the $k+1$ triangles $\triangle(i, i+(k+1), i+2(k+1))$ for $0 \leq i \leq k$. Since $\log_2 n = k + \log_2(3(k+1)) \geq k+2 > k+1$ for $k \geq 1$, then the total number of triangles in this construction is at least

$$\begin{aligned}&\textstyle\sum_{t=1}^{k}(n/2^t - 2t) + (k+1)\\ &= \textstyle\sum_{t=1}^{k}(3(k+1)2^{k-t} - 2t) + (k+1)\\ &= n - (k+2)(k+1) > n - (\log_2 n)^2.\end{aligned}$$

It can be verified that the constructed set of triangles has the local non-overlapping property. □

The next figure shows the set of 88 triangles on 96 points given by the previous construction for $k = 3$. For clarity, every other point and its corresponding triangle formed with its neighboring points are omitted. Triangles and vertices are color-coded according to the stage they are added/removed. For complete detail, the figures can be arbitrarily zoomed-in in the digital version of this note.



To close the results for the convex case, we report the exact values for $f^{conv}(n)$ for n up to 35. These were obtained by mathematical analysis aided by computer search.

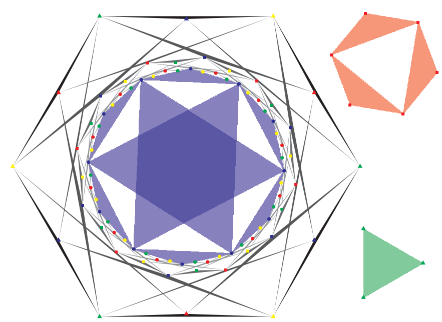
$$f^{conv}(n) = \begin{cases} n-2 & \text{for} \quad n = 3, \\ n-3 & \text{for} \quad 4 \leq n \leq 6, \\ n-4 & \text{for} \quad 7 \leq n \leq 12, \\ n-5 & \text{for} \quad 13 \leq n \leq 19, \\ n-6 & \text{for} \quad 20 \leq n \leq 35. \end{cases}$$

3 Points in general position

Theorem 2. $f(n) \geq 2n - c' \log^3(n)$.

Proof (sketch). Let k be a positive integer and $n = 3 \cdot 2^{k+2} - 12$. We construct a set of at least $n - 6 - 4\sum_{j=0}^{k-1} f^{conv}(3 \cdot 2^j) \geq 2n - c' \log^3(n)$ locally non-overlapping triangles on a set of n points. We carefully put together four copies of an optimal configuration for $f^{conv}(3 \cdot 2^j)$ for each $0 \leq j \leq k-1$, adding several thin triangles between copies. It can be verified that the constructed set of triangles has the local non-overlapping property.

The next figure shows part of this construction for $n = 84$ ($k = 3$). The complete construction consists of all the thin triangles shown in the figure plus 4 copies of each of the unique optimal configurations for $f^{conv}(3)$, $f^{conv}(6)$, and $f^{conv}(12)$ (each vertex pattern shown in the figure corresponds to one of these copies). □



References

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