As every Calculus student (hopefully) knows, the formula

\[ (f(x)g(x))' = f'(x)g(x) \]

is false in general. However it is true for some particular functions. Find two non-constant functions \( f \) and \( g \) that satisfy this formula.

Solution by Humberto Raya.

When we talk about differentiation, it becomes beneficial to talk about exponential functions and their interesting properties.

For this problem consider:

\[
\begin{align*}
  f(x) &= \exp(mx) \\
  g(x) &= \exp(nx),
\end{align*}
\]

for non-zero, non-one numbers \( m \) and \( n \).

Their derivatives would be:

\[
\begin{align*}
  f'(x) &= m \exp(mx) \\
  g'(x) &= n \exp(nx).
\end{align*}
\]

By the Chain Rule we would have that

\[
(f(x)g(x))' = f(x)g'(x) + g(x)f'(x) \\
= \exp(mx) \exp(nx) + \exp(nx) \exp(mx) \\
= (m+n)\exp((m+n)x).
\]

Note that

\[
f'(x)g'(x) = m \exp(mx) n \exp(nx) \\
= mn \exp((m+n)x).
\]

In order for \((f(x)g(x))' = f'(x)g'(x)\) we must have that

\[
(m+n)\exp((m+n)x) = mn \exp((m+n)x)
\]

By dividing both sides by \( \exp((m+n)x) \) we’ll see that we need:

\[ m+n = mn \]

By algebra we have that \( n = mn - m = (n-1)m \) and so \( m = n/(n-1) \).

In particular, let \( n = 2 \) so that \( m = 2 \).

Then \( f(x) = \exp[2x] \) and \( g(x) = \exp[2x] \) and we’ll have what we wanted:

\[
\begin{align*}
  [f(x)g(x)]' &= \exp[2x]2 \exp[2x] + \exp[2x]2 \exp[2x] \\
  &= 2 \exp[4x] + 2 \exp[4x] \\
  &= 4 \exp[4x] \\
  &= [2 \exp[2x]] [2 \exp[2x]] \\
  &= f'(x)g'(x)
\end{align*}
\]