Circles of radii 1 and 2 are externally tangent to each other and internally tangent to a third circle of radius 3. A fourth circle is tangent to the other three circles according to the figure.

What is the radius of this fourth circle?

**Solution 1 (by the organizers).** Let $A, B, C, D$ be the centers of the circles with radii 2,3,1, and $r$ respectively. The problem asks for the value of $r$. The circles with centers $A$ and $C$ are externally tangent and internally tangent to the circle of radius 3, thus $A, B,$ and $C$ are collinear. Note that $AC = 3, AB = 1, BC = 2, AD = 2 + r, CD = 1 + r,$ and $BD = 3 - r$. Since $BC = 2AB$ then $\text{Area}(BCD) = 2\text{Area}(ABD)$. We calculate both of these areas using Heron’s formula: If a triangle has sides $a, b, c,$ and semiperimeter $s = (a + b + c)/2$, then its area equals $\sqrt{s(s-a)(s-b)(s-c)}$.

In $ABD$ the semiperimeter equals $(AB + BD + AD)/2 = 3,$ and in $BCD$ the semiperimeter equals $(BC + CD + BD)/2 = 3.$ Thus

\[
\text{Area}(ABD)^2 = 3\,(2\,r\, (1 - r)) = 6r \,(1 - r), \quad \text{and}
\]

\[
\text{Area}(BCD)^2 = 3\,(1 \,(2 - r) \,r) = 3r \,(2 - r).
\]

Then

\[3r \,(2 - r) = \text{Area}(BCD)^2 = 4\text{Area}(ABD)^2 = 24r \,(1 - r)\]

and since $r \neq 0$ then $r = 6/7$. 

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Solution 2 (by the organizers). We use the same labels as in solution 1, but now we also consider the point \( X \), given as the foot of the perpendicular to \( AC \) by \( D \). Instead of using areas, we use Pythagora’s Theorem applied to the triangles \( AXD, BXD \), and \( CXD \). Let \( DX = h \) and \( CX = a \). Then

\[
\begin{align*}
(1 + r)^2 &= CD^2 = CX^2 + XD^2 = a^2 + h^2, \\
(3 - r)^2 &= BD^2 = BX^2 + XD^2 = (2 - a)^2 + h^2, \text{ and} \\
(r + 2)^2 &= AD^2 = AX^2 + XD^2 = (3 - a)^2 + h^2.
\end{align*}
\]

Solving for \( h^2 \) in the first equation and substituting in the other two we get

\[
\begin{align*}
2r - 1 &= a, \text{ and} \\
3 - r &= 3a.
\end{align*}
\]

Solving for \( a \) and \( r \) we get \( a = 5/7 \) and \( r = 6/7 \).