Let $a$ and $b$ be positive integers such that $a$ divides $b^2$, $b^2$ divides $a^3$, $a^3$ divides $b^4$, $b^4$ divides $a^5$, but $a^5$ does not divide $b^6$. Find with proof a pair $(a, b)$ with this property where $a$ is as small as possible.

Solution (by organizers). The pair $(a, b)$ with smallest $a$ is $(16, 8)$. First note that $a = 2^4 \mid 2^6 = b^2$, $b^2 = 2^6 \mid 2^{12} = a^3$, $a^3 = 2^{12} \mid 2^{12} = b^4$, $b^4 = 2^{12} \mid 2^{20} = b^5$, but $b^5 = 2^{20} \nmid 2^{24} = a^6$.

Now, suppose $(a, b)$ is the pair with smallest $a$ satisfying the conditions $a^3 \mid b^4$ and $a^5 \nmid b^6$. By the Fundamental Theorem of Arithmetic we may assume that $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ and $b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$, where the $p_i$s are distinct primes and $\alpha_i, \beta_i \geq 0$ for $i = 1, 2, \ldots, r$.

The condition $a^3 \mid b^4$ implies that $3\alpha_i \leq 4\beta_i$ for every $i = 1, 2, \ldots, r$. Similarly, $a^5 \nmid b^6$ implies that $5\alpha_j > 6\beta_j$ for some $1 \leq j \leq r$. Let $A = 2^{\alpha_j}$ and $B = 2^{\beta_j}$, observe that $A^3 \mid B^4$, $A^5 \nmid B^6$, and $a \geq A$. Thus we may assume that $a = A = 2^{\alpha_j} = 2^\alpha$ and $b = B = 2^{\beta_j} = 2^\beta$ with $6/5 < \alpha/\beta \leq 4/3$. The smallest numerator $\alpha$ of all fractions $\alpha/\beta$ in the range $(6/5, 4/3]$ is precisely $\alpha = 4$ (with $\beta = 3$). Therefore $(a, b) = (2^4, 2^3) = (16, 8)$ is the pair with smallest $a$ satisfying $a^3 \mid b^4$ and $a^5 \nmid b^6$ and, as we saw before, it also satisfies that $a \mid b^2$, $b^2 \mid a^3$, and $b^4 \mid a^5$. 