Problem of the Week #2

March 15-22, 2004

Original Problem: Find with proof the exact value of the sum

\[ \frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots + \frac{n}{2^n} + \ldots \]

Note: If you are familiar with Σ-notation, the sum above can be written as

\[ \sum_{n=1}^{\infty} \frac{n}{2^n}. \]

Further questions for future research projects.

1. You can modify the problem replacing 2 by 3, namely, find the value of

\[ \sum_{n=1}^{\infty} \frac{n}{3^n}. \]

In general, given a positive integer \( k \), find the value of

\[ \sum_{n=1}^{\infty} \frac{n}{k^n}. \]

2. In (1) we vary the denominator of the terms in the series. This time we vary the numerator. Find the value of

\[ \sum_{n=1}^{\infty} \frac{n^2}{2^n}. \]

In general, given a positive integer \( j \), find the value of

\[ \sum_{n=1}^{\infty} \frac{n^j}{2^n}. \]

3. Put together the previous two questions. Given positive integers \( j \) and \( k \), find the value of

\[ \sum_{n=1}^{\infty} \frac{n^j}{k^n}. \]

Note: Before trying to find the value of any of these series, it is important to prove that they actually converge.