Problem of the Week 9, Fall 2008

Triangle $ABC$ has $AC = 9$, $AB = 12$, and $BC = 15$. The points $M$ and $N$ are the midpoints of the segments $AC$ and $AB$, respectively. A point $L$ is constructed on segment $BC$, such that $LC = 3$.

Segments $BM$ and $CN$ intersect at $O$, and segment $AL$ intersects $BM$ and $CN$ at $P$ and $Q$, respectively. What is the area of triangle $OPQ$?

\[ \begin{array}{c}
A \\
\hspace{1cm} N \\
\hspace{2cm} O \\
\hspace{3cm} P \\
\hspace{4cm} M \\
B \hspace{2cm} L \hspace{1cm} C
\end{array} \]

**Solution by organizers.** Because $9^2 + 12^2 = 15^2$, then $ABC$ is a right triangle. We use coordinate geometry placing $A$ on the origin, $B$ on the positive $x$-axis, and $C$ on the positive $y$-axis. Then, $A = (0,0), B = (12,0), C = (0,9), M = (0,\frac{9}{2}), N = (6,0)$. To find the coordinates $(L_x,L_y)$ of $L$, let $R$ be the point on $AC$ such that $L'R$ is horizontal. Triangles $ABC$ and $RLC$ are similar and thus $\frac{RL}{AB} = \frac{LC}{BC} = \frac{CR}{CA}$. That is, $\frac{L_x}{12} = \frac{3}{15} = \frac{9-L_y}{9}$. This gives $L = (L_x,L_y) = (\frac{12}{5}, \frac{36}{5})$. To find the coordinates of $O, P,$ and $Q$, we find the equations of the lines $AL, BM,$ and $CN$.

\[ \begin{align*}
\overrightarrow{AL} & : y = 3x, \\
\overrightarrow{BM} & : y = -\frac{3}{8}x + \frac{9}{2}, \\
\overrightarrow{CN} & : y = -\frac{3}{2}x + 9.
\end{align*} \]

The three intersections are

\[ \begin{align*}
O & = \overrightarrow{BM} \cap \overrightarrow{CN} = (4,3), \\
P & = \overrightarrow{AL} \cap \overrightarrow{BM} = \left(\frac{4}{3},4\right), \\
Q & = \overrightarrow{AL} \cap \overrightarrow{CN} = (2,6).
\end{align*} \]

Finally, the area of triangle $PQR$ is

\[ \frac{1}{2} \left| \det \begin{pmatrix}
4 & 4 & 2 \\
3 & 4 & 6 \\
1 & 1 & 1
\end{pmatrix} \right| = 3. \]