Problem of the Week 3, Fall 2008

Consider an isosceles triangle $ABC$ with $AB = AC$. Let $D$ be the midpoint of segment $BC$, $E$ a point on the side $AC$ such that $DE$ is perpendicular to $AC$, and $F$ the midpoint of segment $DE$. Prove that the segments $AF$ and $BE$ are perpendicular.

Solution by the organizers. Let $M$ be the midpoint of $AD$. Because $F$ is the midpoint of $ED$, then $MF$ is parallel to $AE$. Hence triangles $MDF$ and $ADE$ are similar. Note that also, triangles $ADE$, $DCE$, and $ABD$ are similar. This means that $\angle DMF = \angle CDE$, and thus so are their supplements $\angle BDE = \angle AMF$. We prove that triangles $BDE$ and $AMF$ are similar by showing that $\frac{BD}{AM} = \frac{DE}{MF}$. Indeed,

$$\frac{BD}{AM} = \frac{BD}{\frac{1}{2}AD} = \frac{DF}{\frac{1}{2}MF} = \frac{DE}{\frac{1}{2}MF} = \frac{DE}{MF},$$

because $AM = \frac{1}{2}AD$, triangles $ABD$ and $MDF$ are similar, and $DF = \frac{1}{2}DE$.

Let $P$ and $Q$ be the intersections of $BE$ with $AD$ and $AF$, respectively. The fact that $BDE$ and $AMF$ are similar triangles implies that $\angle DBP = \angle DBE = \angle MAF = \angle PAQ$. Also $\angle BPD = \angle QPA$, because they are vertical angles. Therefore the remaining angles in triangles $BPD$ and $APQ$ are equal. That is $\angle AQP = \angle PDB = \angle ADB$ which is a right angle.