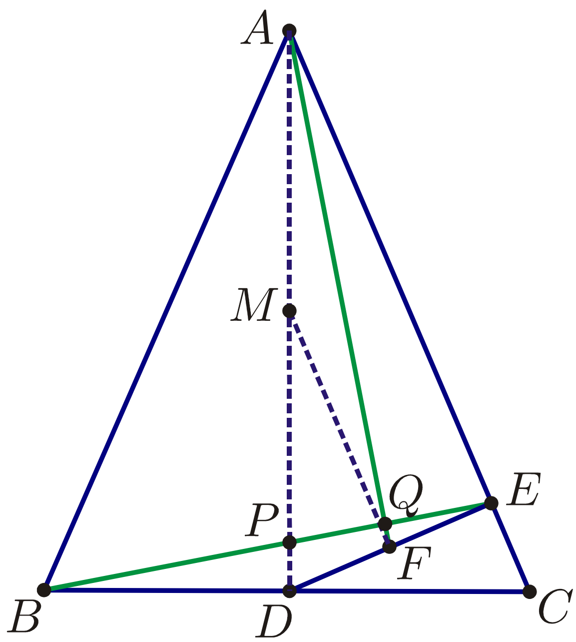
Problem of the Week 3, Fall 2008

Consider an isosceles triangle ABC with $AB = AC$. Let D be the midpoint of segment BC, E a point on the side AC such that DE is perpendicular to AC, and F the midpoint of segment DE. Prove that the segments AF and BE are perpendicular.



Solution by the organizers. Let M be the midpoint of AD. Because F is the midpoint of ED, then MF is parallel to AE. Hence triangles MDF and ADE are similar. Note that also, triangles ADE, DCE, and ABD are similar. This means that $\measuredangle DMF = \measuredangle CDE$, and thus so are their supplements $\measuredangle BDE = \measuredangle AMF$. We prove that triangles BDE and AMF are similar by showing that $\frac{BD}{AM} = \frac{DE}{MF}$. Indeed,

$$\frac{BD}{AM} = \frac{BD}{\frac{1}{2}AD} = \frac{DF}{\frac{1}{2}MF} = \frac{\frac{1}{2}DE}{\frac{1}{2}MF} = \frac{DE}{MF},$$

because $AM = \frac{1}{2}AD$, triangles ABD and MDF are similar, and $DF = \frac{1}{2}DE$.

Let P and Q be the intersections of BE with AD and AF, respectively. The fact that BDE and AMF are similar triangles implies that $\measuredangle DBP = \measuredangle DBE = \measuredangle MAF = \measuredangle PAQ$. Also $\measuredangle BPD = \measuredangle QPA$, because they are vertical angles. Therefore the remaining angles in triangles BPD and APQ are equal. That is $\measuredangle AQP = \measuredangle PDB = \measuredangle ADB$ which is a right angle.