Problem of the Week 5, Fall 2006

Solution by organizers. Suppose that the points \((n, m)\) and \((p, q)\), with \(n, m, p,\) and \(q\) integers, are on a circle with center \((\sqrt{2}, \sqrt{3})\). Then \((\sqrt{2}, \sqrt{3})\) is on the perpendicular bisector of the segment \((n, m) (p, q)\). This perpendicular bisector passes through the midpoint \(\left(\frac{n+p}{2}, \frac{m+q}{2}\right)\) and has slope \(\frac{-1}{(\frac{n+p}{m+q})} = \frac{m-n}{m-q}\). Then the equation of the perpendicular bisector is

\[
y = \frac{p-n}{m-q} \left( x - \frac{n+p}{2} \right) + \frac{m+q}{2}
\]

which can be written as

\[
2(m-q)y = 2x(p-n) + (n^2 - p^2 + m^2 - q^2).
\]

Since the point \((\sqrt{2}, \sqrt{3})\) is on this line then

\[
2(m-q)\sqrt{3} = 2\sqrt{2}(p-n) + (n^2 - p^2 + m^2 - q^2).
\]

Squaring both sides of this equation and isolating \(\sqrt{2}\) we get

\[
12(m-q)^2 = 8(p-n)^2 + (n^2 - p^2 + m^2 - q^2)^2 + 4\sqrt{2}(p-n)(n^2 - p^2 + m^2 - q^2)
\]

\[
\sqrt{2} = \frac{12(m-q)^2 - 8(p-n)^2 - (n^2 - p^2 + m^2 - q^2)^2}{4(p-n)(n^2 - p^2 + m^2 - q^2)}.
\]

Since \(n, m, p,\) and \(q\) are all integers then the right hand side of this identity is a rational number. But we know \(\sqrt{2}\) is irrational, so we have a contradiction.