Problem of the Week 9, Fall 2005

**Solution by Andrew Jones.** First we show 2005 cannot be written as the sum of numbers each of them equal to 119 or 18. That is, it is not possible to write $2005 = 119A + 18B$ where $A$ and $B$ are integers, $A \geq 0$ and $B \geq 0$.

**proof (by contradiction)**
Assume $2005 = 119A + 18B$ for some $A$ and $B$ are integers, $A \geq 0$ and $B \geq 0$. (1)
For (1) to hold we need $A < 16$ because if $A > 17$ then $119A + 18B > 119(17) = 2023 > 2005$.
This means that $A$ equals 0, 1, 2, 3,…, 15, or 16. We can go through all of these possibilities and disprove them all.

For $A = 0$, Equation (1) gives $2005 = 18B$. But $B = 2005/18$ is not an integer. Thus $A \neq 0$.
For $A = 1$, Equation (1) gives $B = 1886/18$, not an integer. Thus $A \neq 1$. Similarly for $A = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$ and 16, Equation (1) gives $B = 1767/18, 1648/18, 1529/18, 1410/18, 1291/18, 1172/18, 1053/18, 934/18, 815/18, 696/18, 577/18, 458/18, 339/18, 220/18$, and $101/18$.
But all these values of $B$ are not integers so Equation (1) never holds.

Now we will prove that for any integer $n \geq 2006$ we can write $n$ in the form $n = 119A + 18B$ where $A$ and $B$ are non-negative integers.

**proof (by Mathematical Induction)**
P(n) : $n = 119A + 18B$ where $A$ and $B$ are integers and $A \geq 0$ and $B \geq 0$.

**Basis Step**
P(2006) is true: $2006 = 119(4) + 18(85)$

**Inductive Step**
We assume P(n) is true for some $n \geq 2006$. That is, $n = 119A + 18B$ for some $A \geq 0$ and $B \geq 0$. (2)
We want to prove that P(n+1) is true. That is, $n+1 = 119C + 18D$ for some $C \geq 0$ and $D \geq 0$. This is done by cases and we will use the fact that we can add and/or subtract 119's and 18's as long as we are left with a positive number of 119's and 18's.

**Case 1**
Note that $1 = 119(5) + 18(-33)$. (3)
If $B \geq 33$ then adding (2) and (3) gives $n+1 = 119(A+5) + 18(B-33)$ where $A+5 \geq 0$ and $B-33 \geq 0$.

**Case 2**
Note that $1 = 119(-13) + 18(86)$. (4)
If $B \leq 32$ then $A \geq 13$ or $A \leq 12$. But $A \leq 12$ and $B \leq 32$ gives $n \leq 119(12) + 18(32) = 2004$ which is a contradiction. Thus $A \geq 13$. Similarly to Case 1, adding (2) and (4) gives $n+1 = 119(A-13) + 18(B+86)$ where $A-13 \geq 0$ and $B+86 \geq 0$.

In both cases we proved P(n+1) to be true. Therefore by Mathematical Induction P(n) is true for all $n \geq 2006$. 