Problem of the Week 13, Fall 2005

Solution by the organizers. The answer is 2499. Any $n \in \mathbb{N}$ can be written as

$$n = 99q_1 + r_1 \text{ and } n = 101q_2 + r_2$$

where q_1, q_2, r_1 and r_2 are non-negative integers such that $0 \le r_1 \le 98$ and $0 \le r_2 \le 100$. If n is a solution to

$$\left\lfloor \frac{n}{99} \right\rfloor = \left\lfloor \frac{n}{101} \right\rfloor$$

then $q_1 = q_2$. Thus

$$n = 99q_1 + r_1 = 101q_1 + r_2$$

and

$$r_1 - r_2 = 2q_1.$$

This means that all pairs of integers r_1 and r_2 that have the same parity (both even or both odd) and such that $98 \ge r_1 \ge r_2 \ge 0$ generate a solution (with the exception of $r_1 = r_2 = 0$ because n cannot be 0). Therefore the possible values for r_1 and r_2 are

r_1	r_2	#
1	1	1
2	2,0	2
3	3,1	2
4	4,2,0	3
5	5,3,1	3
6	6,4,2,0	4
7	7,5,3,1	4
⋮	⋮	
96	96,94,92,...,4,2,0	49
97	97,95,93,...5,3,1	49
98	98,96,94,...,4,2,0	50.

Hence the number of solutions is

$$2\,(1 + 2 + 3 + 4 + ... + 50) - 1 - 50 = 2\left(\frac{50\,(51)}{2}\right) - 1 - 50 = 2499.$$