Problem of the Week 12, Fall 2005

Solution #1 by the organizers. If \( x + y = 0 \) then \( y = -x \) and thus

\[
\left( \sqrt{1+x^2} + x \right) \left( \sqrt{1+y^2} + y \right) = \left( \sqrt{1+x^2} + x \right) \left( \sqrt{1+(-x)^2} + (-x) \right) \\
= \left( \sqrt{1+x^2} + x \right) \left( \sqrt{1+x^2} - x \right) \\
= 1 + x^2 - x^2 = 1.
\]

Now assume that

\[
\left( \sqrt{1+x^2} + x \right) \left( \sqrt{1+y^2} + y \right) = 1.
\]

Multiplying both sides by \( \sqrt{1+y^2} - y \)

\[
\left( \sqrt{1+x^2} + x \right) \left( \sqrt{1+y^2} + y \right) \left( \sqrt{1+y^2} - y \right) = \sqrt{1+y^2} - y \\
\sqrt{1+x^2} + x = \sqrt{1+y^2} - y.
\]

This gives

\[
x + y = \sqrt{1+y^2} - \sqrt{1+x^2}.
\]

Now square both sides and simplify

\[
x^2 + y^2 + 2xy = 1 + y^2 + 1 + x^2 - 2\sqrt{1+y^2}\sqrt{1+x^2} \\
\sqrt{(1+y^2)(1+x^2)} = 1 - xy.
\]

Finally square both sides again and regroup

\[
1 + y^2 + x^2 + y^2x^2 = 1 - 2xy + x^2y^2 \\
y^2 + x^2 + 2xy = 0 \\
(x+y)^2 = 0 \\
x + y = 0.
\]
Solution #2 by the organizers. Follow the previous proof up to

\[ \sqrt{1 + x^2} + x = \sqrt{1 + y^2} - y. \]  \hfill (1)

Let \( f(x) = \sqrt{1 + x^2} + x \). Then the derivative of \( f \) satisfies

\[ f'(x) = \frac{x}{\sqrt{1 + x^2}} + 1 = \frac{x + \sqrt{1 + x^2}}{\sqrt{1 + x^2}} > \frac{x + |x|}{\sqrt{1 + x^2}} \geq 0. \]

Since \( f'(x) > 0 \) for all \( x \) then \( f \) is strictly increasing and hence one-to-one. Since \( f(x) = f(-y) \) from (1) then \( x = -y \), that is, \( x + y = 0 \).