Problem of the Week 6, Fall 2005

Solution by the organizers. Assume first that $a \leq b \leq c$. Since each of $a$, $b$, and $c$ divides $a + b + c$ then $a$ divides $b + c$, $b$ divides $a + c$, and $c$ divides $a + b$. Note that $0 < a + b \leq 2c$ and since $c$ divides $a + b$ then either $a + b = c$ or $a + b = 2c$. Moreover $a + b = 2c$ if and only if $a = b = c$. This gives the solution $(a, a, a)$ for all positive integers $a$ (note that in fact $a$ divides $a + a + a = 3a$).

Now assume $a + b = c$. Since $b$ divides $a + c = 2a + b$, then $b$ divides $2a$. But $0 < 2a \leq 2b$ so $2a = b$ or $2a = 2b$. If $2a = b$ then the triple $(a, 2a, 3a)$ is a solution for all positive integers $a$ (note that $a$ divides $a + 2a + 3a = 6a$). If $2a = 2b$, then the triple $(a, a, 2a)$ is a solution for all positive integers $a$ (note that $a$ divides $a + a + 2a = 4a$).

Therefore all the solution triples are:

$$(a, b, c) \in \{(n, n, n), (n, 2n, 3n), (n, n, 2n) : n \in \mathbb{N}\}.$$