Let $ABC$ be an isosceles triangle with $BC = CA$ and $\angle BCA = 20^\circ$. Points $D$ and $E$ are on the sides $BC$ and $CA$, respectively, and they satisfy that $\angle DAB = 50^\circ$ and $\angle ABE = 60^\circ$. Find, with proof, the exact value of the angle $\angle DEB$.

**Solution by:** Yuko Takagi

* This figure may not be accurate.
Objectives: To prove \( \triangle CAD \) and \( \triangle BED \) are similar triangles by comparing the ratio of the lengths of the sides, and find out the value of the angle \( \angle DEB \).

[Step 1]
From the given information, the followings are determined.
\( \angle ADB = 50^\circ \)
\( \angle CBE = 20^\circ \)
\( \angle DAC = 30^\circ \)

[Step 2]
Let the lengths of CA and CB = 1.
Let the lengths of AB = a.

From the given information and Step 1, \( \triangle BAD \) is an isosceles triangle.
Therefore, the length of DB = a.

[Step 3]
Let the length of CE equal to x.
Since \( \triangle CEB \) is an isosceles triangle (see Step 1), the length of BE = x.
Now, I describe x in terms of a.

[Step 4]
Draw a perpendicular line from C to the line AB. Label it F.
Draw a perpendicular line from E to the line AB. Label it G.

\( \frac{FG}{GA} = \frac{CE}{EA} = \frac{x}{1-x} \rightarrow (1) \)

[Step 5]
Since \( \triangle ABC \) is an isosceles triangle, F equally divides the line AB.
\( \triangle BEG \) is a half of a equilateral triangle, therefore the length of BG is half of the length of BE.

\( \frac{FG}{GA} = \frac{BG-BF}{AB-BG} = \frac{x/2 - a/2}{a - x/2} \rightarrow (2) \)

(1) and (2) are the same ratio. Now solve this for x.
\[ x : 1-x = \frac{x}{2} - \frac{a}{2} : a - \frac{x}{2} \]
\[ x = \frac{a}{1-a} \quad \Rightarrow (3) \]

[Step 6]
With those being solved, the ratio of \( BE : BD \) will be
\[ BE : BD = \frac{a}{1-a} : a = 1 : 1-a \quad \Rightarrow (4) \]
which is the same as \( CA : CD \).
Since the ratio of lengths of two sides are the same and the angles between the sides are the same as well, \( \triangle CAD \) and \( \triangle BED \) are similar triangles.
Therefore, \( \angle DEB = \angle DAC = 30^\circ \quad \angle DEB = 30^\circ \)