Suppose $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with $a, b, c, d$ real numbers and $A^3 = \begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix}$. Find $a, b, c, d$.

Note: $A^3$ represents the matrix multiplication of $A$ with itself three times.

Solution (by the organizers). We not only find $a, b, c, d$ but we actually prove there is a unique solution. First we observe that the characteristic polynomial of $A^3$ is
\[ ch_{A^3}(x) = \det \begin{vmatrix} x - 6 & 2 \\ 7 & x - 1 \end{vmatrix} = x^2 - 7x - 8 = (x - 8)(x + 1). \]
We know then that $(A^3 - 8I)(A^3 + I) = 0$. Thus the minimal polynomial $m_A(x)$ of the matrix $A$ should divide the polynomial
\[ (x^3 - 8)(x^3 + 1) = (x - 2)(x + 1)(x^2 - x + 1)(x^2 + 2x + 4) \]
(The quadratic terms above cannot be factored over the reals).

Now, the degree of $m_A$ is not one, otherwise $A$ would be a multiple of the identity, and then clearly $A^3$ would not be equal to $\begin{pmatrix} 6 & 2 \\ 7 & 1 \end{pmatrix}$. Thus the degree of $m_A$ is two. Since $m_A$ has real coefficients then, either $m_A(x) = (x - 2)(x + 1)$, or $m_A(x) = x^2 - x + 1$, or $m_A(x) = x^2 + 2x + 4$. If $m_A(x) = x^2 - x + 1$ then $m_A(A) = A^2 - A + I = 0$, and $0 = (A - I)(A^2 - A + I) = A^3 + I$ which is a contradiction. Similarly, if $m_A(x) = x^2 + 2x + 4$ then $0 = (A - 2I)(A^2 + 2A + 4I) = A^3 - 8I$ which is also a contradiction. Therefore $m_A(x) = ch_A(x) = (x - 2)(x + 1)$ and then $-1$ and $2$ are the two eigenvalues of $A$.

Let $v$ be an eigenvector of $A$ with eigenvalue $-1$. Since $Av = -v$ then $A^3v = -v$, so $v$ is also an eigenvector of $A^3$ with eigenvalue $-1$. Similarly if $u$ is an eigenvector of $A$ with eigenvalue $2$ then $u$ is also an eigenvector of $A^3$ with eigenvalue $8$. It is easy to see that $v = \begin{pmatrix} -2 \\ 7 \end{pmatrix}$ and $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ work as some of these eigenvectors of $A^3$ (others can be used as well). Then since $Av = -v$ and $Au = 2u$ we get the system of equations
\[ -2a + 7b = 2, \quad -2c + 7d = 7, \quad a + b = 2, \quad c + d = 2 \]
which yields the unique solution $a = 4/3, b = 2/3, c = 7/3, \text{ and } d = -1/3.$