Math 140
Introductory Statistics

Professor Silvia Fernández

Chapter 8
Based on the book *Statistics in Action*
by A. Watkins, R. Scheaffer, and G. Cobb.

8.1 Estimating a Proportion with Confidence

- A recent Phi Delta Kappa/Gallup poll reported that a record 51% of the American public assigns a grade of A or B to the public schools in their community and that this survey had a margin of error of 3%. Source: 2001, www.gallup.com/poll/releases/pr010823.asp.

- These results are based on telephone interviews with a randomly selected national sample of 1108 adults, 18 years and older, conducted May 23–June 6, 2001.

- For results based on this sample, one can say with 95 percent confidence that the maximum error attributable to sampling and other random effects is plus or minus 3 percentage points. In addition to sampling error, question wording and practical difficulties in conducting surveys can introduce error or bias into the findings of public opinion polls.

8.1 Estimating a Proportion with Confidence

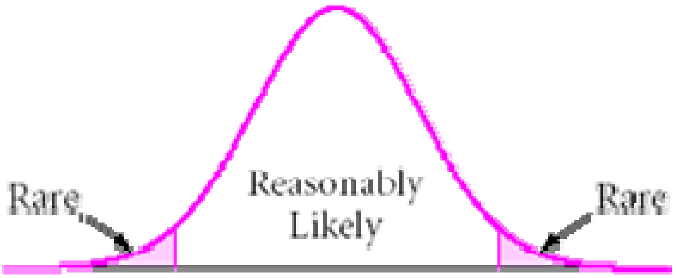
- A recent Phi Delta Kappa/Gallup poll reported that a record 51% of the American public assigns a grade of A or B to the public schools in their community and that this survey had a margin of error of 3%. Source: 2001, www.gallup.com/poll/releases/pr010823.asp.

- The Gallup organization is disclosing that they didn't ask all adults in the United States, only 1108. Even so, unless there are some special difficulties such as problems with the wording of the question, they are 95% confident that the error is less than 3% either way in the percentages they report. That is, they are 95% confident that if they were to ask all adults in the United States to give a grade to the public schools, 51% ± 3%, or between 48% and 54%, would give a grade of A or B. How can the Gallup organization possibly make such a statement?

Reasonably Likely (Again)

- We learned in 7.3 that if we get a sample of size n from a population with proportion of success p, then the **reasonably likely** outcomes fall between the values

$$p \pm 1.96\sqrt{\frac{p(1-p)}{n}}$$



- Recall that reasonably likely outcomes are those in the middle 95% of the distribution of all possible outcomes. The outcomes in the upper 2.5% and the lower 2.5% of the distribution are rare events—they happen, but rarely.

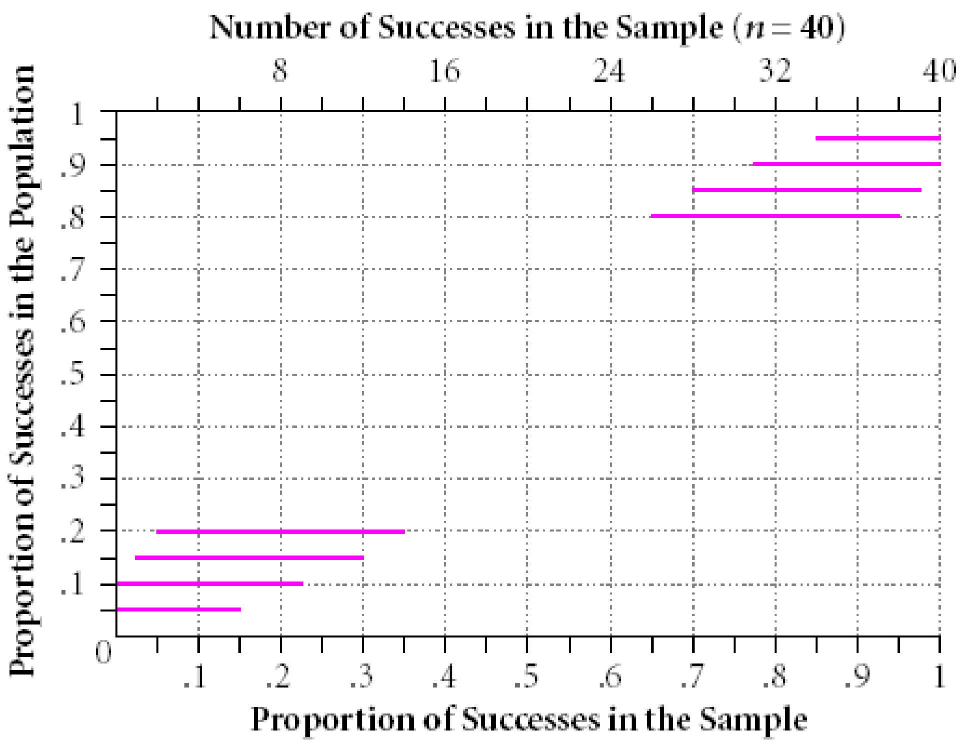
Examples

- Example p.468. Suppose that you will flip a fair coin 100 times. What are the reasonably likely values of the sample proportion $\hat{p}$? What numbers of heads are reasonably likely?

- D1. Suppose 35% of a population think they pay too much for car insurance. A polling organization takes a random sample of 500 people in this population and computes the sample proportion of people who think that they pay too much for car insurance.
  - a. There is a 95% chance that $\hat{p}$ will be between what two values?
  - b. Is it reasonably likely to get 145 people in the sample who think they pay too much for car insurance?

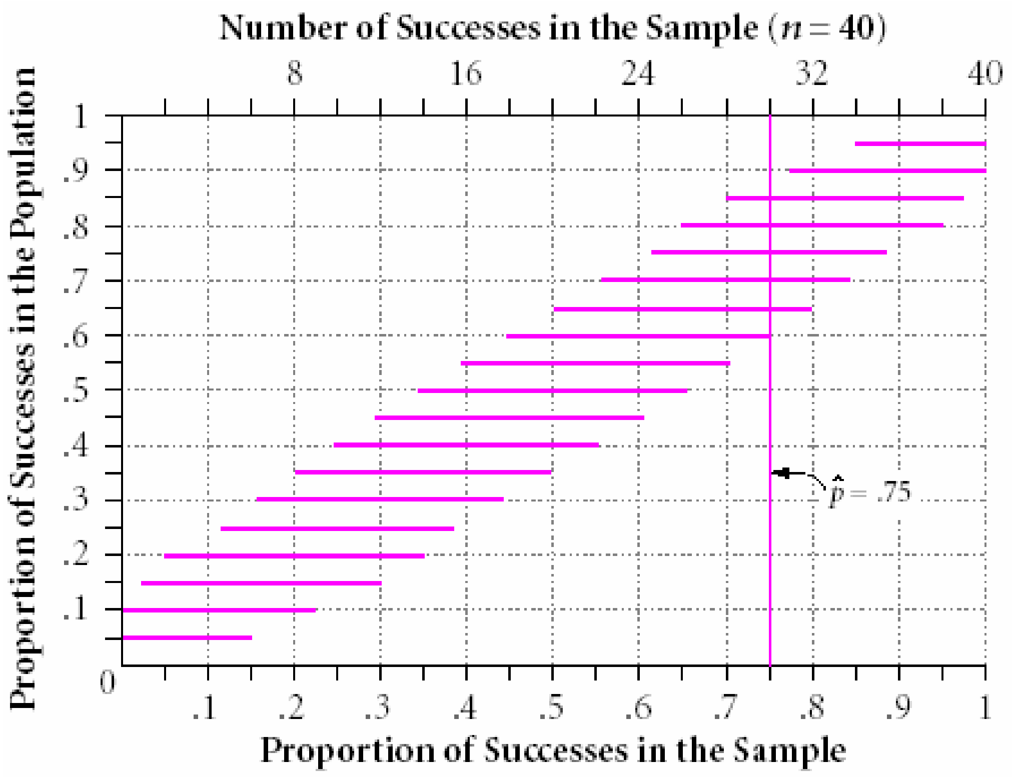
Activity 8.1b

- 1. Your instructor will give you one of the population proportions whose line segments are missing in Display 8.1.
- 2. Compute the reasonably likely outcomes for your population proportion p.
- 3. On your copy of the chart in Display 8.1, draw a horizontal line segment showing the reasonably likely outcomes for your group's proportion p.
- 4. Get the reasonably likely outcomes from the other groups in your class and complete the chart with the line segments from those values of p.



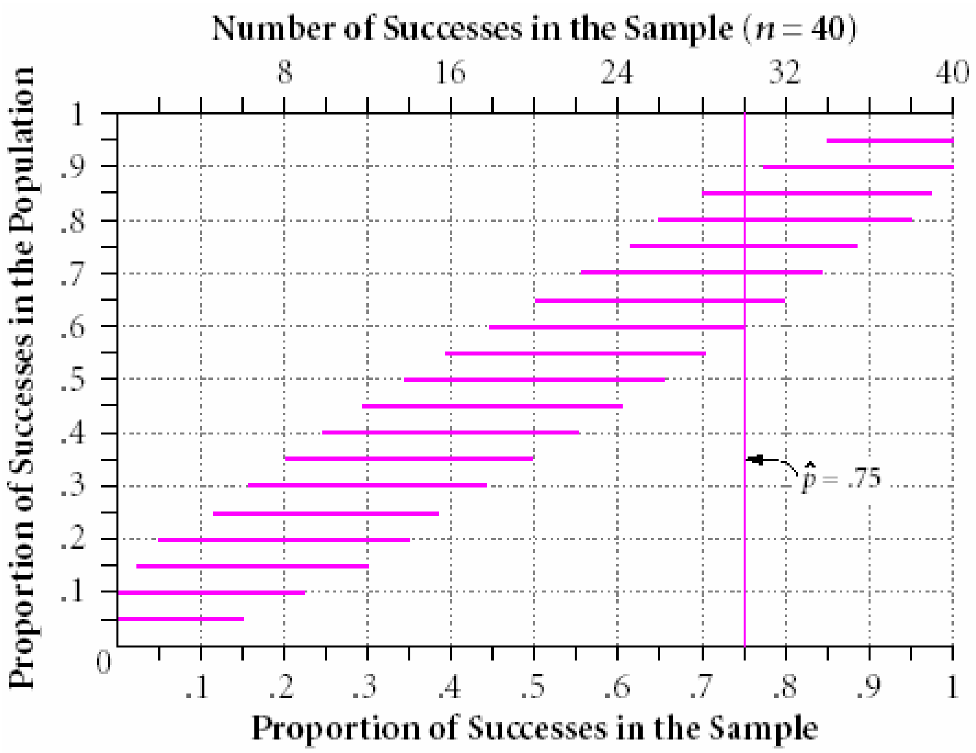
Activity 8.2 Results

p	Left	Right
0.25	0.1158	0.3841
0.3	0.1579	0.4420
0.35	0.2021	0.4978
0.4	0.2481	0.5518
0.45	0.2958	0.6041
0.5	0.3450	0.6549
0.55	0.3958	0.7041
0.6	0.4481	0.7518
0.65	0.5021	0.7978
0.7	0.5579	0.8420
0.75	0.6158	0.8841



Example: Plausible Percentages

- In a group of 40 adults, exactly 30 were right-eye dominant. Assuming this can be considered a random sample of all adults, is it plausible that if you tested *all* adults, you would find that 50% are right-eye dominant? Is it plausible that 80% are right-eye dominant? What percentages are plausible?



Confidence Interval

- A **95% confidence interval** consists of those population percentages p for which the sample proportion $\hat{p}$ is reasonably likely.
- Notes:
  - In the previous diagram the horizontal segments were reasonably likely intervals.
  - The 95% confidence intervals should be represented as vertical segments. In this case p is the unknown parameter. In the previous example the 95% confidence interval goes from 0.6 to about 0.85

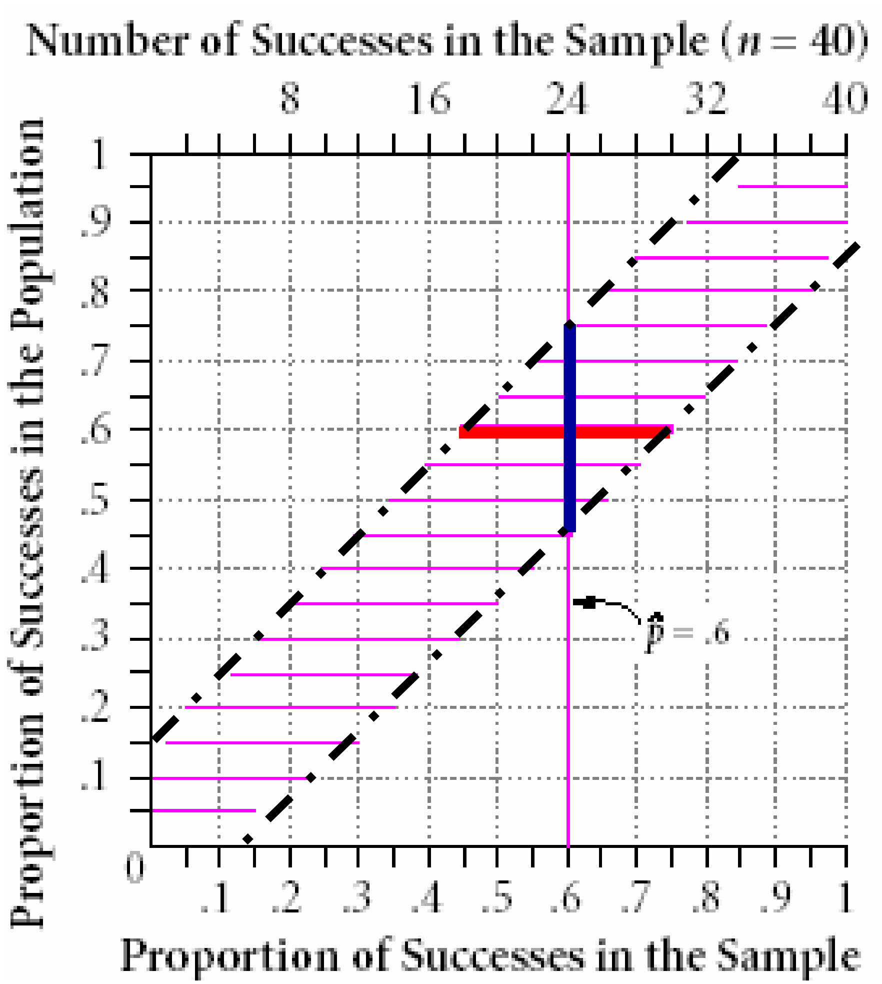
Discussion: Confidence Intervals

- D3. According to the 2000 U.S. Census, about 60% of Hispanics in the United States are of Mexican origin. Would it be reasonably likely in a survey of 40 randomly chosen Hispanics to find that 27 are of Mexican origin?

  Source: www.census.gov/prod/2001pubs/c2kbr01-3.pdf.
- D4. According to the 2000 U.S. Census, about 30% of people over age 85 are men. In a random sample of 40 people over age 85, would it be reasonably likely to get 60% who are men?

  Source: www.census.gov/prod/2001pubs/c2kbr01-10.pdf.
- D5. Suppose that in a random sample of 40 toddlers, 34 know what color Elmo is. What is the 95% confidence interval for the percentage of toddlers who know what color Elmo is?
- D6. Polls usually report a margin of error. Suppose a poll of 40 randomly selected statistics majors finds that 20 are female. The poll reports that 50% of statistics majors are female, with a margin of error of 15%. Use your completed chart to explain where the 15% came from.

Getting a Formula



- The endpoints of the horizontal segment (a reasonably likely interval) are:

$$p \pm 1.96\sqrt{\frac{p(1-p)}{n}}$$

- The horizontal segments are delimited by two curves that are almost parallel lines of slope 1.
- Thus the vertical segment has about the same length as the horizontal segment.
- Moreover, since the center of each horizontal segment is p, then the endpoints of the vertical segment are:

$$\hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

A Confidence Interval for a Proportion. (Any percent)

- A **confidence interval** for the proportion of successes p in the population is given by the formula

$$\hat{p} \pm z \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- n is the sample size, $\hat{p}$ is the proportion of successes in the sample. The value of z depends on how confident you want to be that $\hat{p}$ will be in the confidence interval. For a 95% confidence interval, use z = 1.96; for a 90% confidence interval, use z = 1.645; for a 99% confidence interval, use z = 2.576; and so on.

- This confidence interval is reasonably accurate when **three conditions** are met:
  - The sample was a simple random sample from a binomial population (every subject is either a success or a failure).
  - Both $n\hat{p}$ and $n(1-\hat{p})$ are at least 10.
  - The size of the population is at least 10 times the size of the sample n.

A Confidence Interval and the Margin of Error

- A confidence interval for the proportion of successes p in the population is given by the formula

$$\hat{p} \pm z \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$z = \begin{cases} 1.645 & \text{for 90\% confidence} \\ 1.96 & \text{for 95\% confidence} \\ 2.576 & \text{for 99\% confidence} \end{cases}$$

- This confidence interval is reasonably accurate when three conditions are met:
  - The sample was a SRS from a binomial population (every subject is either a success or a failure).
  - $n\hat{p} \geq 10$ and $n(1-\hat{p}) \geq 10$.
  - The size of the population is at least 10 times the size of the sample n.

- The quantity

$$E = z \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

is called **the margin of error**. It is half the length of the confidence interval.

Example: Safety Violations.

- Suppose you have a random sample of 40 buses from a large city and find that 24 have a safety violation. Find the 90% confidence interval for the proportion of all buses that have a safety violation.

Activity 8.3 (Simulated)

1. Generate a random sample of 40 numbers between 0 and 9.
2. Count the number of even digits in your sample of 40.
3. Construct a 95% confidence interval for the proportion of random digits that are even.
4. Repeat a 100 times and draw all of the intervals in the appropriate display (like Display 8.5 in p. 426)
5. What is the true proportion of all random digits that are even?
6. What percentage of the confidence intervals captured the true proportion? Is this what you expected? Explain.

- Examples:
  1 6 9 6 3 0 9 1 3 1 2 8 3 5 6 0 0
  0 6 7 8 3 1 4 9 6 5 6 9 6 7 5 3 2
  6 8 1 4 1 2
  Even: 20 $\hat{p} = \frac{20}{40} = 0.5$
  95% Confidence Interval
  $$0.5 \pm (1.96)\sqrt{\frac{0.5(1-0.5)}{40}} =$$
  (0.34505, 0.65495)
- 8 9 4 4 2 6 6 8 7 4 9 6 3 4 8 8 2
  7 4 4 2 0 3 6 1 6 0 5 2 0 9 8 2 7
  2 1 7 2 5 8
  Even: 27 $\hat{p} = \frac{27}{40} = 0.675$
  95% Confidence Interval
  $$0.675 \pm (1.96)\sqrt{\frac{0.675(1-0.675)}{40}} =$$
  (0.52985, 0.82015)

Activity 8.3 (Simulated)

- Examples:
  1 6 9 6 3 0 9 1 3 1 2 8 3 5 6 0 0
  0 6 7 8 3 1 4 9 6 5 6 9 6 7 5 3 2
  6 8 1 4 1 2
  Even: 20 $\hat{p} = \frac{20}{40} = 0.5$
  95% Confidence Interval
  $$0.5 \pm (1.96)\sqrt{\frac{0.5(1-0.5)}{40}} =$$
  (0.34505, 0.65495)
- 8 9 4 4 2 6 6 8 7 4 9 6 3 4 8 8 2
  7 4 4 2 0 3 6 1 6 0 5 2 0 9 8 2 7
  2 1 7 2 5 8
  Even: 27 $\hat{p} = \frac{27}{40} = 0.675$
  95% Confidence Interval
  $$0.675 \pm (1.96)\sqrt{\frac{0.675(1-0.675)}{40}} =$$
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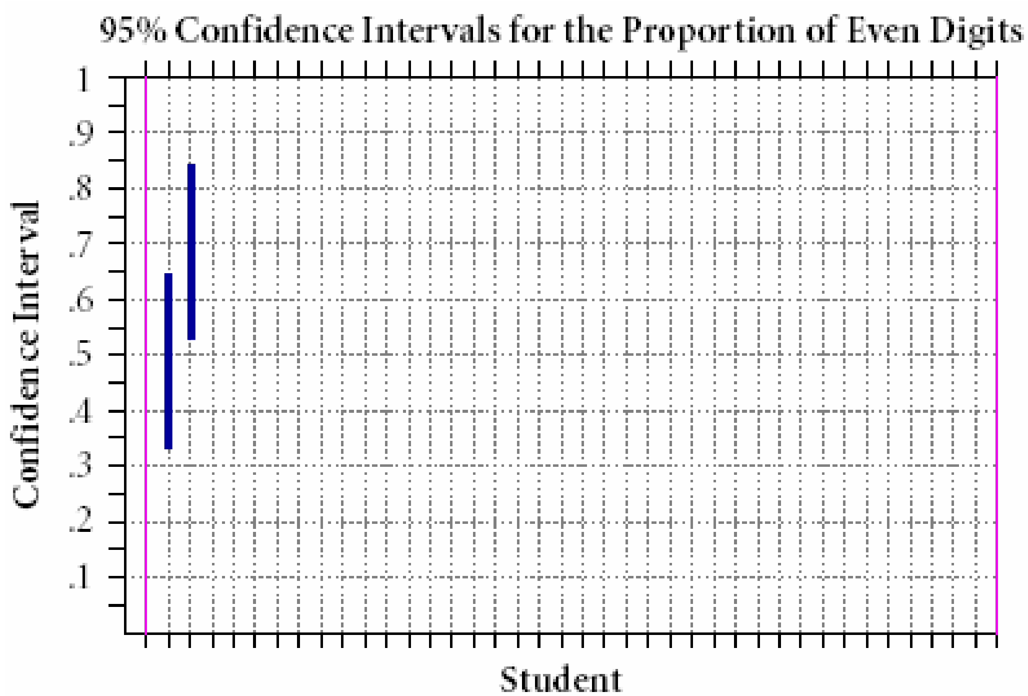
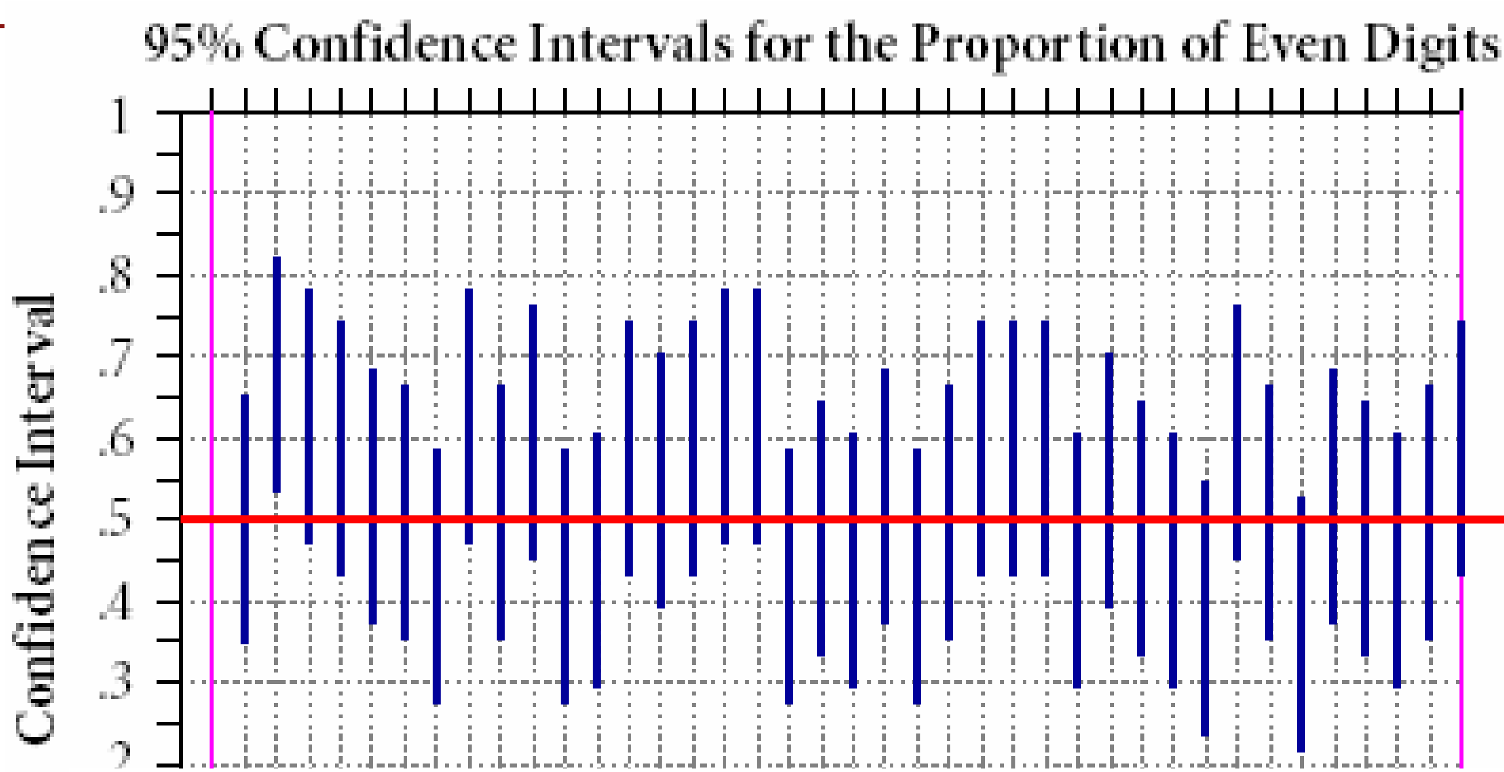


Activity 8.3 (Simulated)



How many intervals contain the true proportion of even numbers p = 0.5?

Answer: all except one, that is 38 out of 39. Or 97.43% of all of the intervals.

In general, if we randomly calculate a large number of 95% Confidence Intervals we should expect that about 95% of them will contain the true value of p.

Back to Opinion Polls.

- At the beginning of this section, you read about a recent Phi Delta Kappa/Gallup poll that reported that a record 51% of the American public assigns a grade of A or B to the public schools in their community. The sample size was 1108. This survey had a margin of error of 3%, and so their 95% confidence interval is 48% to 54%. (The procedure Gallup uses to select a sample is more complicated than simple random sampling, but you can use your formula for a confidence interval to approximate Gallup's margin of error.)

You now should be able to answer these questions:

- What is it that you are 95% sure is in the confidence interval?

  Answer: The proportion of *all* Americans who would assign a grade of A or B to their local public schools.

Back to Opinion Polls.

- At the beginning of this section, you read about a recent Phi Delta Kappa/Gallup poll that reported that a record 51% of the American public assigns a grade of A or B to the public schools in their community. The sample size was 1108. This survey had a margin of error of 3%, and so their 95% confidence interval is 48% to 54%. (The procedure Gallup uses to select a sample is more complicated than simple random sampling, but you can use your formula for a confidence interval to approximate Gallup's margin of error.)

You now should be able to answer these questions:

- What is the interpretation of the confidence interval of 48% to 54%?

  Answer: We are 95% confident that if we could ask all Americans to give a grade to their local public schools, between 48% and 54% of them would give an A or B.

Back to Opinion Polls.

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You now should be able to answer these questions:

- What is the meaning of "95% confidence"?

  Answer: If we were to take 100 random samples of Americans and compute the 95% confidence interval from each sample, then we expect that 95 of them will contain the true proportion of all Americans that would assign a grade of A or B (whatever that proportion is).

What sample size should you use?

- Example. (p.426) Suppose you take a survey and get $\hat{p} = 0.7$. If your sample size is 100, what would be the margin of error for a 95% confidence interval?

$$E = z \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = (1.96)\sqrt{\frac{0.7(1-0.7)}{100}} = .0898$$

- If you quadruple your sample, what would be the new margin of error?

$$E = z \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = (1.96)\sqrt{\frac{0.7(1-0.7)}{400}} = .0449$$

What sample size should you use?

- Simple Answer: In general, the larger the sample the more accurate the results will be (smaller margin of error). If we would like to find the sample size for a particular margin of error, all we have to do is solve for n.

$$E = z \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$E^2 = z^2 \cdot \left(\frac{\hat{p}(1-\hat{p})}{n}\right)$$

$$n = \frac{z^2 \cdot \hat{p}(1-\hat{p})}{E^2}$$

If you do not have an estimate for $\hat{p}$, then you should use 0.5.

You may get a larger n than you need but never a smaller one.

Example: Estimating needed sample sizes (p.481)

- What sample size should you use for a survey if you want the margin of error to be at most 3% with 95% confidence but you have no estimate of p?

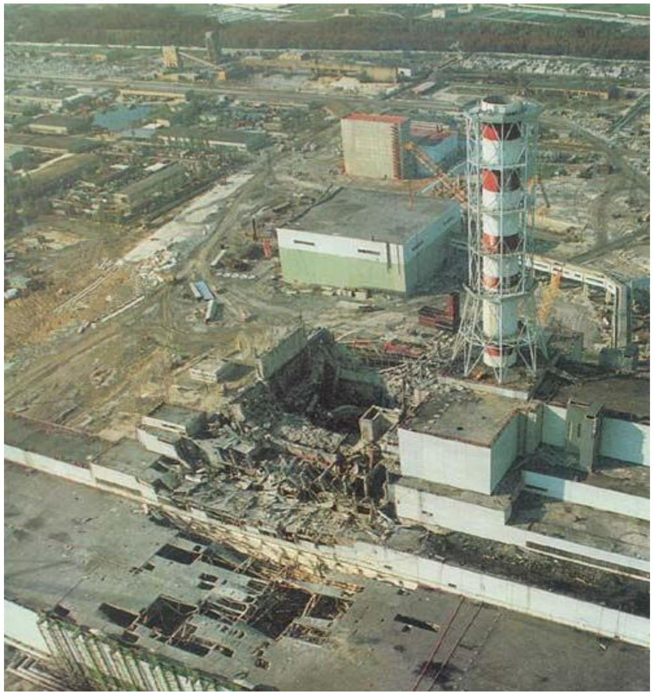
- D17. Suppose it costs $1 to survey each person in your sample. You judge that p is about 0.5. What will your survey cost if you want a margin of error of about 10%? 1%? 0.1%?

8.2 Testing a Proportion

- People often make decisions with data by comparing the results from a sample to some predetermined standard. These kinds of decisions are called **tests of significance.**
- Goal**:** To test the significance of the difference between the sample and the standard.
  - Small difference: there is no reason to conclude that the standard doesn't hold.
  - Large enough difference: If it can't reasonably be attributed to chance, you can conclude that the standard no longer holds.

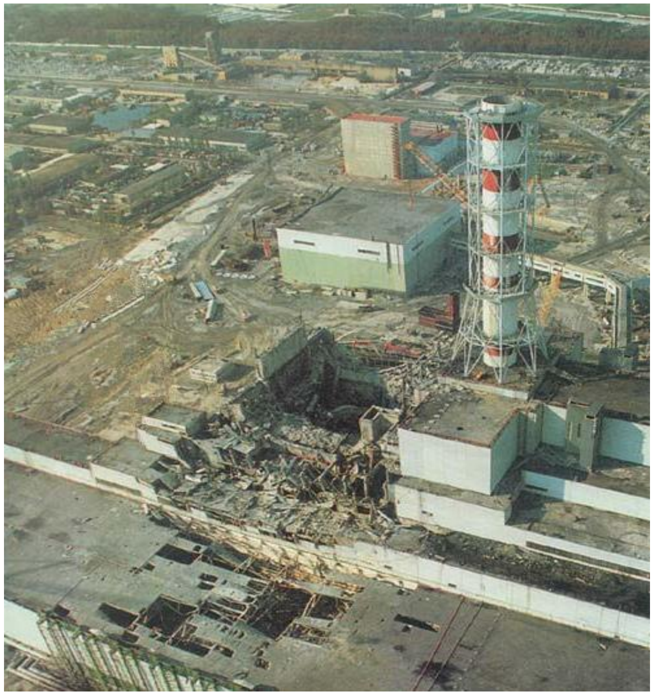
Example

- About 2% of barn swallows have white feathers in places where the plumage is normally blue or red. The white feathers are caused by genetic mutations.
- In 1986, the Russian nuclear reactor at Chernobyl leaked radioactivity. Researchers continue to be concerned that the radiation may have caused mutations in the genes of humans and animals that were passed on to offspring.



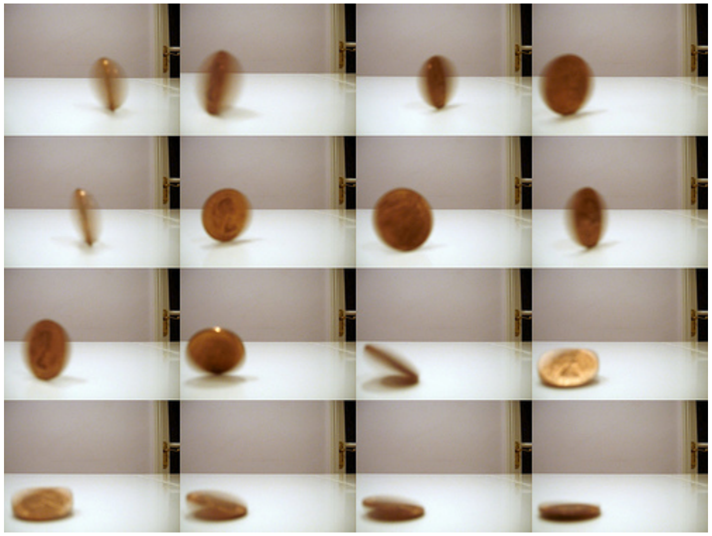
Example

- In a sample of barn swallows captured around Chernobyl in 1991 and 1996, about 14% had white feathers in places where the plumage is normally blue or red.
- Researchers compared the proportion 0.14 in the sample of captured barn swallows to the standard of .02. If the overall percentage was still only 2%, **it is not reasonably likely** to get 14% in their sample.
- So they came to the conclusion that there was an increased probability of genetic mutations in the Chernobyl area. Source: *Los Angeles Times*, October 9, 1997, page B2.



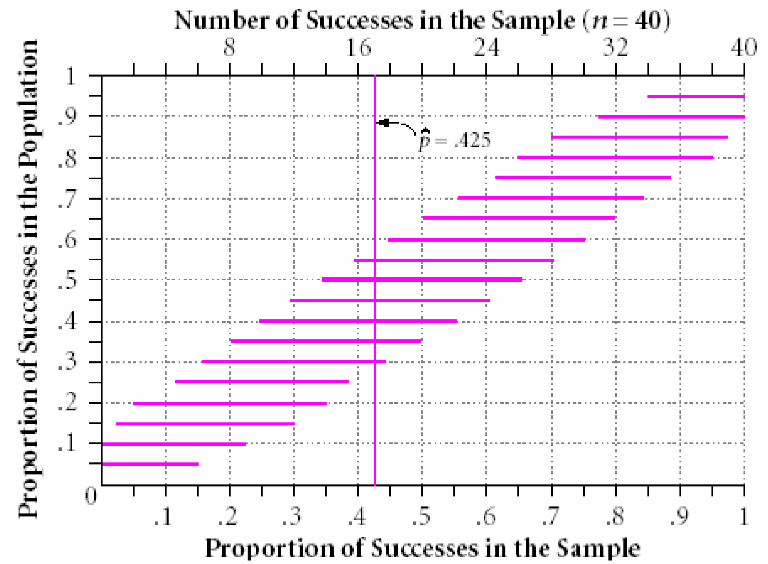
Informal Significance Testing

- People tend to believe that pennies are balanced. They generally have no qualms about flipping a penny to make a fair decision. Is it really the case that penny flipping is fair? What about spinning pennies?
- The logic involved in deciding whether or not to reject the standard that spinning a penny results in heads 50% of the time makes use of the same logic as that involved in estimating a proportion in Section 8.1.



Spinning Pennies
Jenny and Maya's Spins

- Jenny and Maya wonder if heads and tails are equally likely when a penny is spun. They spin pennies 40 times and get 17 heads. Should they reject the standard that pennies fall heads 50% of the time even if heads and tails are equally likely?



$$\hat{p} = \frac{17}{40} = 0.425$$

The value 0.425 falls in the reasonably likely interval obtained from p = 0.5.

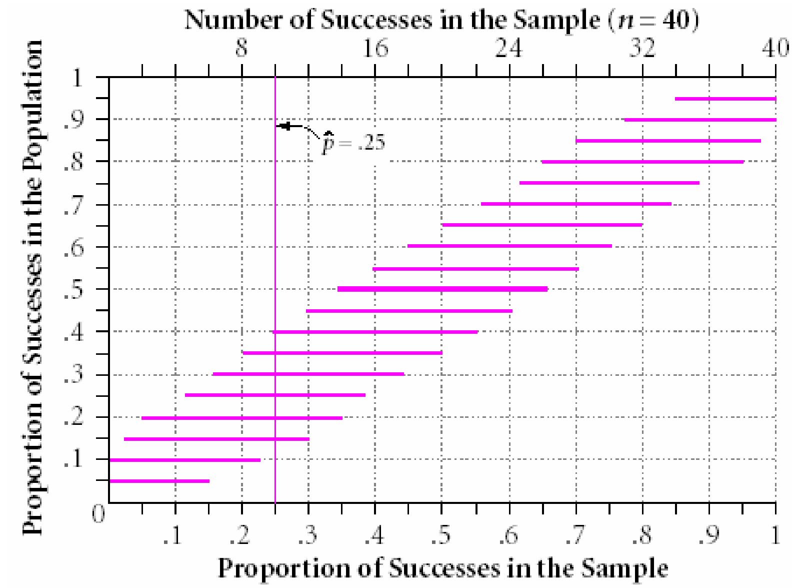
Statistical Significance

- A sample is **statistically significant** if it is **not a reasonably likely outcome** when the proposed standard is true.
- Jenny and Maya's result is not statistically significant since their sample proportion of 0.425 falls within the reasonably likely interval of p = 0.5.

Spinning Pennies
Miguel and Kevin's Spins

- Miguel and Kevin also spun pennies and got 10 heads out of 40 spins for a sample proportion of 0.25. Is this a statistically significant result?

$$\hat{p} = \frac{10}{40} = 0.25$$



The value 0.25 falls outside the reasonably likely interval obtained from p = 0.5.

This *is* a statistically significant result!

Spinning Pennies
Miguel and Kevin's Spins

- Miguel and Kevin also spun pennies and got 10 heads out of 40 spins for a sample proportion of 0.25. Is this a statistically significant result?

$$\hat{p} = \frac{10}{40} = 0.25$$

- The value 0.25 falls outside the reasonably likely interval obtained from p = 0.5.

- Another solution is to calculate the 95% Confidence Interval using $\hat{p} = 0.25$ and verify that 0.5 is not in it.

$$\hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} =$$

$$0.25 \pm 1.96\sqrt{\frac{(0.25)(1-0.25)}{40}} =$$

$$= (0.11581, 0.38419)$$

Basic Notation.

p Population proportion of successes
(Unknown in general)

$\hat{p}$ Sample proportion of successes
(What we recorded from our sample)

p_0 Hypothesized value of the population proportion. (The Standard)

Discussion: Statistical Significance

- A 1997 article reported that two-thirds of teens in grades 7–12 want to study more about medical research. You wonder if this proportion still holds today and decide to test it. You take a random sample of 40 teens and find that only 23 want to study more about medical research.

  Source: CNN Interactive Story Page, www.cnn.com/tech/9704/22/teentech.poll/, April 22, 1997.

  - a. What is the standard (the hypothesized value, p_0, of the population proportion)?
  - b. What is an alternate hypothesis?
  - c. What is the sample proportion?
  - d. Is the result statistically significant? That is, is there evidence leading you to believe that the proportion today is different from the proportion in 1997?

Discussion: Statistical Significance

- a. What is the standard (the hypothesized value, p_0, of the population proportion)?

  Answer: p_0 = 2/3 = 66.66%

- b. What is an alternate hypothesis?

  Answer: That the proportion nowadays is different from that in 1997, that is that p_0 is different from 2/3.

- c. What is the sample proportion?

  Answer: The sample proportion is

$$\hat{p} = 23/40 = 0.575$$

Discussion: Statistical Significance

- d. Is the result statistically significant? That is, is there evidence leading you to believe that the proportion today is different from the proportion in 1997?
- One possible solution is to calculate the 95% Confidence Interval and then check whether the value p_0 = 2/3 is in the interval or not.

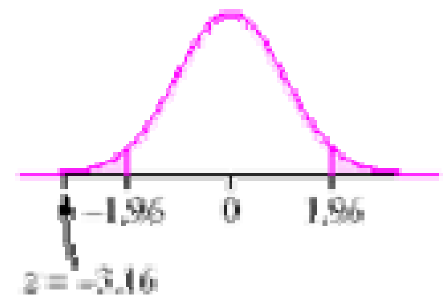
Do it!

- Here is the standard procedure.

The Test Statistic

- To check if the sample proportion $\hat{p}$ is statistically significant with respect to the standard p_0 we just need to check if $\hat{p}$ is a rare event in the distribution generated by p_0.
- We know the distribution is approximately normal with

$$\mu = p_0, \text{ and } \sigma = \sqrt{\frac{p_0(1-p_0)}{n}}$$



- So we can calculate the z-score of $\hat{p}$:

$$z = \frac{\hat{p}-\mu}{\sigma} = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

This value of z is called the Test Statistic

- And if we get z < –1.96 or z > 1.96 then $\hat{p}$ is statistically significant.
- The numbers –1.96 and 1.96 are called **critical values**.

Other Critical Values

- The dividing points are called **critical values** (denoted z*). Other z*-values commonly used as critical points are

$$z^* = \begin{cases} 1.645 & \text{for a level of significance of } \alpha = 10\% \\ 1.96 & \text{for a level of significance of } \alpha = 5\% \\ 2.576 & \text{for a level of significance of } \alpha = 1\% \end{cases}$$

- If the value of the test statistic is more extreme than the critical values you have chosen, you reject the standard and say that the result is statistically significant.

- A larger critical value makes it harder to reject the standard. If you use $z^* = \pm 1.96$, then to reject the standard, the test statistic z must fall in the outer 5% of the standard normal distribution. If you use $z^* = \pm 1.645$, the value of z must fall in only the outer 10% of the distribution.

- Each critical value is associated with a corresponding percentage, α (alpha), called the **level of significance.** If a level of significance isn't specified, it is usually safe to assume that $\alpha = .05$ and $z^* = \pm 1.96$.

Example

- Find the test statistic from Jenny and Maya's data on spinning pennies. (17 Heads out of 40 spins) What do you conclude if α = 0.10?
- Recall that the test statistic is:

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1 - p_0)}{n}}}$$

- Since we are testing if the probability of getting heads is 0.5, then we have p_0 = 0.5, n =40, and sample proportion p hat of 17/40 = 0.425. Thus

$$z = \frac{0.425 - 0.5}{\sqrt{\frac{0.5(1 - 0.5)}{40}}} = -0.948683$$

The critical values associated to α = 0.10 are z^* = ±1.645.

Since our test statistic falls in between, then the conclusion is that the sample is **not** statistically significant.

Example

- Use your *z*-table (or your calculator preferably) to answer these questions.
  - a. What level of significance is associated with critical values of z^* = ±2.576?

Answer: We need to find the area under the curve between –2.576 and 2.576 in the standard normal distribution. We can do this using:

$$\text{normalcdf}(-2.576\ ,\ 2.576) = 0.9900048$$

So the level of significance is equal to 100% – 99% = 1%.

  - b. What critical values are associated with a level of significance of 2%?

Answer: We need to find the z-values that correspond to the middle 98% of the area under the standard normal distribution, that is the values that leave 1% at the beginning and 1% at the end. By symmetry we can find only one of them. We can do this using:

$$\text{invNorm}(0.01) = -2.32634$$

Formal Language of Test Significance

(Components of a Significance Test for a Proportion)

1. **Give the name of the test and check the conditions for its use.** For a significance test for a proportion, three conditions must be met.

- The sample is a simple random sample from a binomial population.
- Both np_0 and $n(1 – p_0)$ are at least 10.
- The population size is at least 10 times the sample size.

Formal Language of Test Significance

(Components of a Significance Test for a Proportion)

2. **State the hypotheses, defining any symbols.**
When testing a proportion, the null hypothesis H_0 is
   - H_0 : The percentage of successes p in the population from which the sample came is equal to p_0.
- The alternate hypothesis, Ha, can be of three forms:
   - H_a : The percentage of successes p in the population from which the sample came is not equal to p_0.
   - H_a : The percentage of successes p in the population from which the sample came is greater than p_0.
   - H_a : The percentage of successes p in the population from which the sample came is less than p_0.

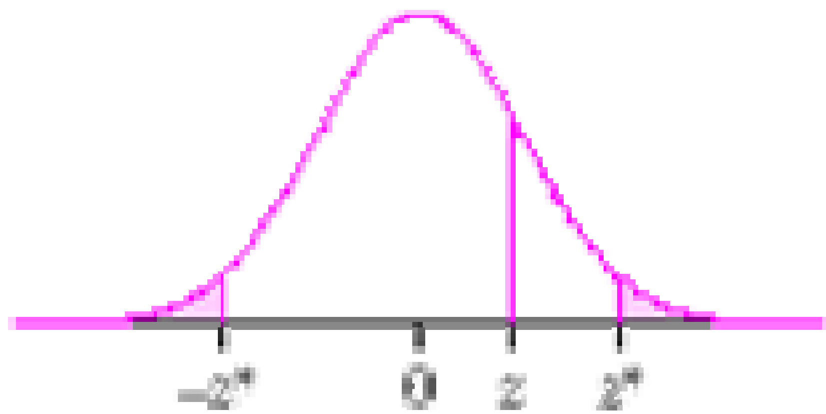
Formal Language of Test Significance

(Components of a Significance Test for a Proportion)

3. **Compute the test statistic z and compare it to the critical values z^*** (or find the P-value—as explained later in this section).

- The test statistic is

$$z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$



- Compare the value of z to the predetermined critical values. Include a sketch that illustrates the situation.

Formal Language of Test Significance

(Components of a Significance Test for a Proportion)

4. **Write a conclusion.** There are two parts to stating a conclusion:

- Say whether you reject the null hypothesis or don't reject the null hypothesis, linking your reason to the results of your computations.
- Tell what your conclusion means in the context of the situation.

Note: You should never say that you accept the null hypothesis, because all other values in the confidence interval for p could be plausible values, you cannot assert that p_0 is the right one.

Example
(Similar to example on p. 499.)

- Do the whole analysis for Miguel and Kevin's spinning of 40 pennies getting 10 of them heads.
- That is conduct a significance test to see if a penny comes up heads 50% of the time when spun.

Types of Error

- There are two possible types of error in significance testing:

Null hypothesis is actually

Your decision		True	False
	Don't Reject H_0	Correct	**Type II error**
	Reject H_0	**Type I error**	Correct

Type I error (When Test Statistic is Large)

- If the test statistic is large in absolute value (like Miguel and Kevin's example), then the possible explanations for this are:
  - 1. The null hypothesis is true and a rare event occurred. That is, it was just bad luck that resulted in being so far from p_0.
  - 2. The null hypothesis isn't true, and that's why the sample proportion was so far from p_0.
  - 3. The sampling process was biased in some way, and so the sample value is itself suspicious.
- If the last explanation is ruled out, then the usual decision is to reject the null hypothesis H_0. However, you may be making a **Type I error**—rejecting H_0 even though H_0 is actually true.

Type II error (When Test Statistic is Small)

- If the test statistic is small in absolute value (like Jenny and Maya's sample), then the possible explanations for this are:
  - 1. The null hypothesis is true, and you got just about what you would expect in the sample.
  - 2. The null hypothesis isn't true, and it was just by chance that turned out to be close to p_0.
  - 3. The sampling process was biased in some way, and so the sample value is itself suspicious.
- If the last explanation is ruled out, then the usual decision is to not reject the null hypothesis H_0. However, you may be making a **Type II error**—not rejecting H_0 even though H_0 is actually false.

Minimizing the Error

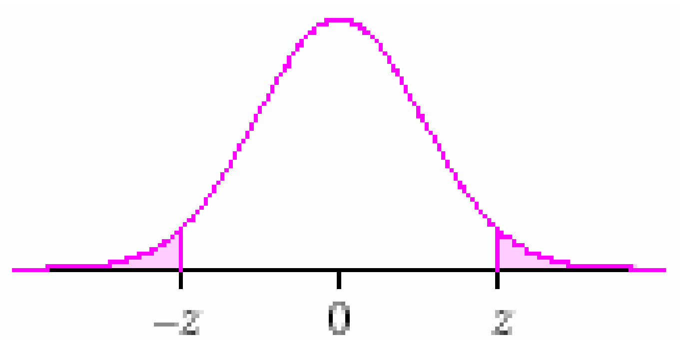
- **Type I Error.** Null hypothesis is true but you reject it.
  If the null hypothesis is true, then the probability of making a Type I error is equal to the significance level of the test. To decrease the probability of a Type I error, decrease the significance level. Changing the sample size has no effect on the probability of a Type I error.

- **Type II Error.** Null hypothesis is false and you fail to reject it.
  To decrease the probability of making a Type II error, you can take a larger sample n or you can increase the significance level α. (But if you do the last option you will increase the probability of a Type I error)

P-Values

- Instead of just reporting that you either have or have not rejected the null hypothesis, it has become common practice also to report a P-value.
- The **P-value** for a test is the probability of seeing a result from a random sample that is as extreme as or more extreme than the one you got from your random sample *if the null hypothesis is true*.

(The P-value for a test is a *conditional probability*)



Example

- Suppose that 22 students out of a random sample of 40 students carry a backpack to school. Follow steps a–d to test the claim that exactly half of the students in the school carry backpacks to class.
  - a. Name the test and check the conditions needed for it.
  - b. State the hypotheses in words and symbols.
  - c. Calculate the value of the test statistic. Calculate the *P*-value for the test. Use this *P*-value in a sentence that explains what it represents.
  - d. What is your conclusion? Explain in the context of this problem.

One-Tailed Tests of Significance

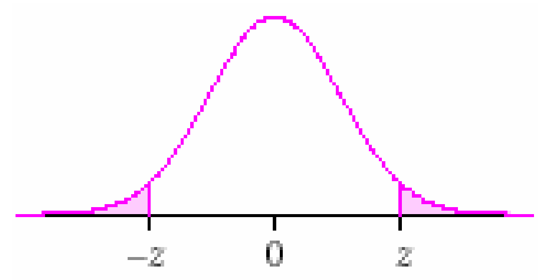
- When testing the effectiveness of a new drug, the investigator must establish that the new drug has a *better* cure rate than the older treatment (or that there are ***fewer*** side effects). He or she isn't interested in simply rejecting the null hypothesis that the new drug has the same cure rate as the older treatment. He or she needs to know if it is ***better***. In such situations, the alternate hypothesis should state that the new drug cures a larger proportion of people than does the older treatment.

- This is called a **one-tailed test of significance.** Tests of significance can be one-tailed if the investigator has an indication of which way any change from the standard should go. This must be decided before looking at the data.

Alternate Hypothesis

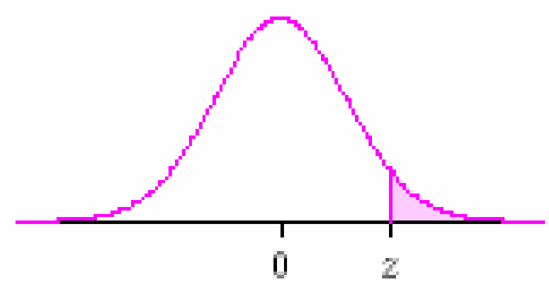
- When testing a proportion, the alternate hypothesis can take one of three forms.
  - H_a : The percentage of successes p in the population from which the sample came is not equal to p_0.

$H_a : p \neq p_0$



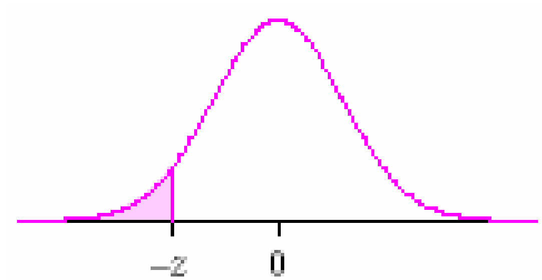
  - H_a : The percentage of successes p in the population from which the sample came is greater than p_0.

$H_a : p > p_0$



  - H_a : The percentage of successes p in the population from which the sample came is less than p_0.

$H_a : p < p_0$



Example: One-Sided Test of Significance

- The editors of a magazine have noticed that people seem to believe that a successful life depends on having good friends. They would like to have a story about this and use a headline such as “Most adults believe friends are important for success.” So they commissioned a survey to ask a random sample of adults whether a successful life depends on having good friends. In a random sample of 1027 adults, 53% said yes. Should the editors go ahead and use their headline?

Example: One-Sided Test of Significance

1. **Give the name of the test and check the conditions for its use.**
   Name: Significance test for a proportion.

n = 1027, $p_0 = 0.5$ (the standard, if having a succesful life is not affected by having good friends or not)

- The sample is a simple random sample (it says in the problem), from a binomial population (a person either agrees or not that a successful life depends on good friends) .
- $np_0 = (1027)(0.5) = 513.5 > 10$
  $n(1 - p_0) = (1027)(1 - 0.5) =$
  $513.5 > 10$
- Total number of adults
  $> 10\ (1027) = 10270$

2. **State the hypotheses, defining any symbols.**

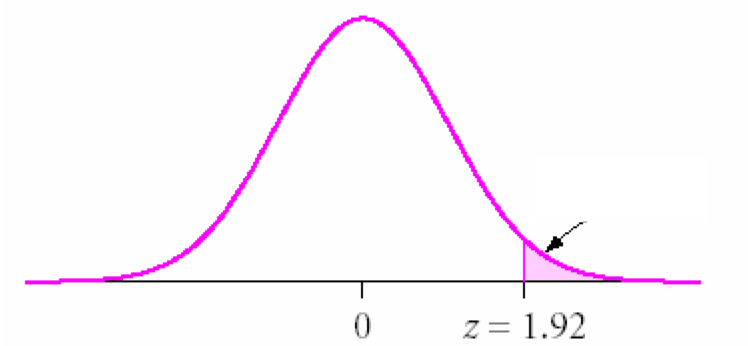
- H_0 : The proportion of people p who believe that a successful life depends on good friends is equal to $p_0 = 0.5$. ($p = p_0$)
- H_a : The proportion of people p who believe that a successful life depends on good friends is greater than $p_0 = 0.5$ ($p > p_0$)

Example: One-Sided Test of Significance

3. Compute the test statistic z and find the P-value

The test statistic is

$$z = \frac{0.53 - 0.5}{\sqrt{\frac{0.5(1-0.5)}{1027}}} = 1.92$$



And then the P-value is the probability that we get a sample with proportion greater than this test statistic (1.92)
(Because we have a one-sided significance test).

Thus P-value = $\mathrm{normalcdf}$(1.92, 999999)
= 0.0274288

Example: One-Sided Test of Significance

4. Write a conclusion.

- Since the P-value equals 0.0274 and this is less than α = 0.05 = 5%. Then we should reject the null hypothesis.

- If the percentage of all adults who believe a successful life depends on having good friends is 50% or less, then the probability of getting a sample proportion of 53% or more is only .0274. Since this value is too small this gives strong evidence that the true percentage is greater than 50%. The editors should feel free to run the headline.

8.3 A Confidence Interval for the Difference of two Proportions

- A very common and important situation involves taking two samples independently from two different populations with the goal of estimating the size of the difference between the proportion of successes in one population and the proportion of successes in the other.

Example

- A recent poll of 29,700 U.S. households found that 63% owned a pet. The percentage in 1994 was 56%. [*Source:* American Pet Products Manufacturers Association, www.appma.org.]
- The two populations are the households in the United States in 1994 and the households now. The question you will investigate is:
- "What was the change in the percentage of U.S. households that own a pet?"

Intuitive Formula

- A confidence interval for the difference of two proportions, $p_1 - p_2$, where p_1 is the proportion of successes in the first population and p_2 is the proportion of successes in the second population:

$$(\widehat{p_1} - \widehat{p_2}) \pm z^* \cdot \text{standard error of}(\widehat{p_1} - \widehat{p_2})$$

  $\widehat{p_1}$ and $\widehat{p_2}$ are the proportions of successes in the two samples.

- In our example, the 95% confidence interval for the difference between the proportion of U.S. households that own pets now and the proportion that owned pets in 1994:

$$(0.63 - 0.56) \pm 1.96 \cdot \text{standard error of}(\widehat{p_1} - \widehat{p_2})$$

Standard Error of the Difference

- The standard errors of the distributions of $\widehat{p}_1$ and $\widehat{p}_2$ can be estimated respectively as:

$$\sqrt{\frac{\widehat{p}_1(1-\widehat{p}_1)}{n_1}} \text{ and } \sqrt{\frac{\widehat{p}_2(1-\widehat{p}_2)}{n_2}}$$

- And if we assume the proportion samples are *independent*, then the standard error of the difference can be approximated as

$$\sqrt{\frac{\widehat{p}_1(1-\widehat{p}_1)}{n_1}+\frac{\widehat{p}_2(1-\widehat{p}_2)}{n_2}}$$

The Formula

- The confidence interval for the difference, $p_1 - p_2$, of the proportion of successes in one population and the proportion of successes in the second population is,

$$(\widehat{p_1} - \widehat{p_2}) \pm z^* \cdot \sqrt{\frac{\widehat{p_1}(1-\widehat{p_1})}{n_1} + \frac{\widehat{p_2}(1-\widehat{p_2})}{n_2}}$$

- Where $\widehat{p_1}$ is the proportion of successes in a random sample of size n_1 taken from the first population, and $\widehat{p_2}$ is the proportion of successes in a random sample of size n_2 taken from the second population. (Sample sizes do not need to be equal.)

Conditions for use

- The conditions that must be met in order to use this formula are that
  - the two samples are taken randomly and independently from two populations.
  - each population is at least 10 times as large as its sample size.
  - $n_1\widehat{p}_1$, $n_1(1-\widehat{p}_1)$, $n_2\widehat{p}_2$, and $n_2(1-\widehat{p}_2)$ are all at least 5.

Example:
A Difference in Pet Ownership?

- A recent pet ownership survey found that 63% of the 29,700 U.S. households sampled own a pet. A 1994 survey, taken by the same organization, found that 56% of the 6,786 U.S. households sampled owned a pet. Find and interpret a 95% confidence interval for the difference between the proportion of U.S. households that own a pet now and the proportion of U.S. households that owned a pet in 1994.

8.4/8.5 Significance Test for the Difference of two Proportions

- Often times we need to decide which is the greater of two proportions, or whether we can assume they are the same. For example.
  - Is snowboarding or skiing more likely to result in a serious injury?
  - Does a new treatment for AIDS result in fewer deaths than an old treatment?
  - Is Reggie Jackson’s World Series record so much better than his play during the regular season that the difference can’t reasonably be attributed to chance?

Example: Two AIDS treatments

- Consider a clinical trial experiment comparing two treatments for AIDS-related complex (ARC). The investigators want to find out if there is a significant difference on the survival rates of patients who had already developed AIDS. They have two treatments, patients who were given AZT and patients who were given AZT + ACV. Here's the data

Treated with

Survived?

	AZT	AZT + ACV	Total
No	28	13	41
Yes	41	49	90
Total	69	62	131

Notation: Proportions, Sample Proportions, and Sample Sizes.

$n_1 = 69$ sample size of patients treated with AZT

$n_2 = 62$ sample size of patients treated with AZT+ACV

$\hat{p}_1 = \frac{41}{69}$ proportion of patients in the sample treated with AZT that survived

$\hat{p}_2 = \frac{49}{62}$ proportion of patients in the sample treated with AZT+ACV that survived

p_1 True proportion of survival if all patients were treated with AZT (unknown)

p_2 True proportion of survival if all patients were treated with AZT+ACV (unknown)

Assumptions about the difference of two proportions.

- If we have obtain two **independent** sample proportions $\hat{p}_1$ and $\hat{p}_2$, then the distribution of the difference of the two proportions $\hat{p}_1 - \hat{p}_2$ is **approximately normal** as long as each proportion satisfies the following three conditions:
  - Each sample $\hat{p}_1$ and $\hat{p}_2$ is a simple random sample from a binomial population, and they are independent from each other.

    or, in case of experiments, subjects were randomly assigned to their treatments.
  - All the numbers $n_1\hat{p}_1, n_2\hat{p}_2, n_1(1-\hat{p}_1), n_2(1-\hat{p}_2)$

    are at least 5.
  - Each of the two population sizes is at least 10 times the sample size.

Checking the conditions

- Check the conditions.
  - We assume that subjects were randomly assigned to treatments.
  - All of the following are greater than 5

$$n_1\hat{p}_1 = 69(41/69) = 41, n_2\hat{p}_2 = 62(49/62) = 49$$

$$n_1(1-\hat{p}_1) = 69(1-41/69) = 28$$

$$n_2(1-\hat{p}_2) = 62(1-49/62) = 13$$

  - The population size of AIDS patients that could potentially be treated with AZT is clearly greater than 10 times 69 = 690

    The population size of AIDS patients that could potentially be treated with AZT+ACV is clearly greater than 10 times 62 = 620

Writing the hypothesis

- The null hypothesis
  - H_0 : The new therapy (AZ+ACV) is as good as the old therapy (AZT). That is the survival rate if all patients were treated with AZT+ACV would be equal to the survival rate if all patients were treated with AZT. In symbols $p_1 = p_2$ or $p_1 - p_2 = 0$
- The alternate hypothesis
  - H_a : The new therapy (AZ+ACV) is better than the old therapy (AZT). That is the survival rate if all patients were treated with AZT+ACV would be greater than the survival rate if all patients were treated with AZT. In symbols $p_1 < p_2$ or $p_1 - p_2 < 0$

The Test Statistic

- The test statistic in general follows the form

$$\text{Test Statistic} = \frac{\text{estimate} - \text{parameter}}{\text{standard deviation of estimate}}$$

- The parameter is $p_1 - p_2$, and if the null hypothesis is true then $p_1 - p_2 = 0$.
- The estimate is what we obtain for the difference of the two proportions according to our samples.

$$\begin{aligned} \text{estimate} &= \hat{p}_1 - \hat{p}_2 \\ &= 0.594 - 0.790 \\ &= -0.196 \end{aligned}$$

- Now, the standard deviation of the estimate under the assumption that $p_1 = p_2$ can be approximated by

$$\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

- Where $\hat{p}$ is the **pooled estimate**

$$\hat{p} = \frac{\text{total survived}}{\text{grand total}} = \frac{90}{131} \approx 0.687$$

The Test Statistic

$$\text{Test Statistic} = \frac{\text{estimate} - \text{parameter}}{\text{standard deviation of estimate}}$$

- The parameter is $p_1 - p_2 = 0$,

$$\begin{aligned}\text{estimate} &= \hat{p}_1 - \hat{p}_2 \\ &= 0.594 - 0.790 \\ &= -0.196\end{aligned}$$

- The standard deviation of the estimate

$$\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

- $\hat{p}$ is the **pooled estimate**

$$\hat{p} = \frac{\text{total survived}}{\text{grand total}} = \frac{90}{131} \approx 0.687$$

- so the standard deviation of the estimate is:

$$\sqrt{(0.687)(1-0.687)\left(\frac{1}{69} + \frac{1}{62}\right)}$$

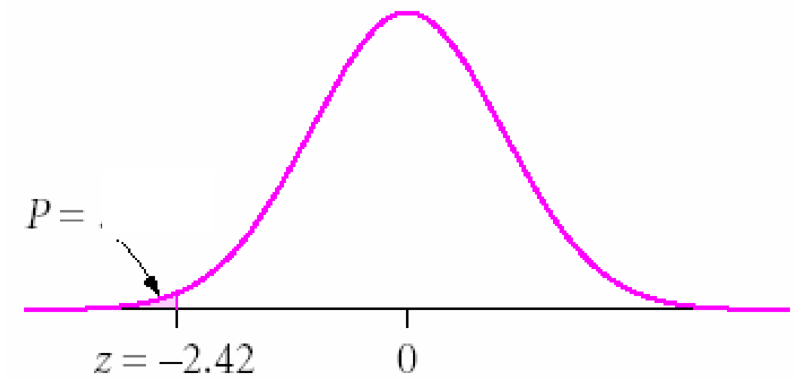
$$= 0.081145$$

- and the test statistic:

$$\frac{-0.196 - 0}{0.081145} = -2.415429$$

P-Value

- To get the P-Value we use the fact that the distribution of the difference is approximately normal.



- Since our test is one-sided we need to calculate the probability that the difference is *less than* our test-statistic.
- We can find P by doing

`normalcdf`(– 999999,– 2.4154)=.007858

Conclusion

- Since the P-Value equals 0.78% and this is definitely less than 5% we reject the null hypothesis.

- If both treatments were equally effective (null hypothesis is true) then there is only a 0.78% chance of getting a difference $\hat{p}_1 - \hat{p}_2$ as small or smaller than – 0.196. This probability is so small that we are confident that if all subjects in the experiment had been given AZT+ACV there would be a larger survival rate than if they had received only AZT.

Components of a Significance Test for the Difference of two Proportions

- See pages 531-532 in your book.

Example E65 (page 537)

- A poll of 256 boys and 257 girls age 12 to 17 asked, "Do you feel like you are personally making a positive difference in your community?" More girls (195) than boys (161) answered "yes."
  - a. Using a one-sided test, is this a statistically significant difference? That is, if all teens were asked, are you confident that a larger proportion of girls than boys would say "yes"? Assume that the samples were selected randomly.
  - b. The report says, "Participants were selected through random digit dialing." Do you have any concerns about whether such a procedure would give a random sample?
  - c. Find a 95% confidence interval for the proportion of all teens who would answer yes. What additional assumption do you need to make to do this?