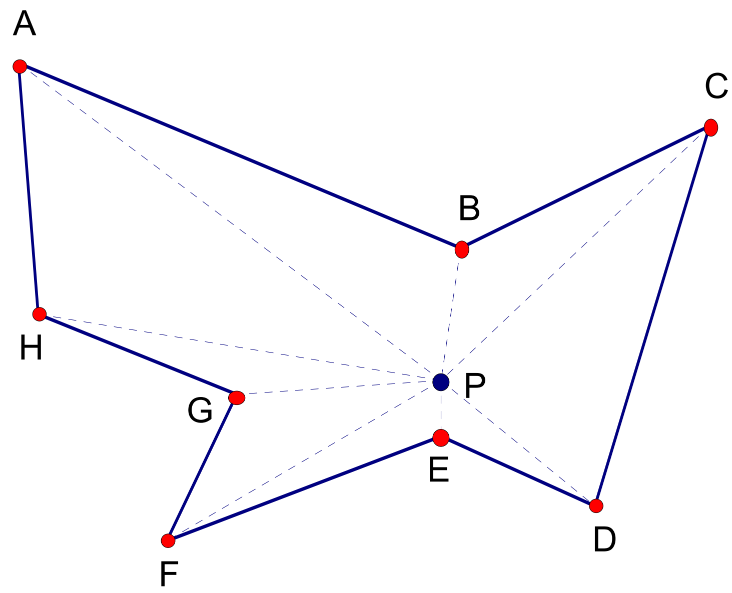
Art Gallery Problems

Prof. Silvia Fernández

Star-shaped Polygons

- A polygon is called *star-shaped* if there is a point *P* in its interior such that the segment joining *P* to any boundary point is completely contained in the polygon.



Krasnosel'skiĭ's Theorem

- Consider a painting gallery whose walls are completely hung with pictures.

  If for each three paintings of the gallery there is a point from which all three can be seen,

  then there exists a point from which all paintings can be seen.

Krasnosel’skiĭ’s Theorem

- Let S be a simple polygon such that

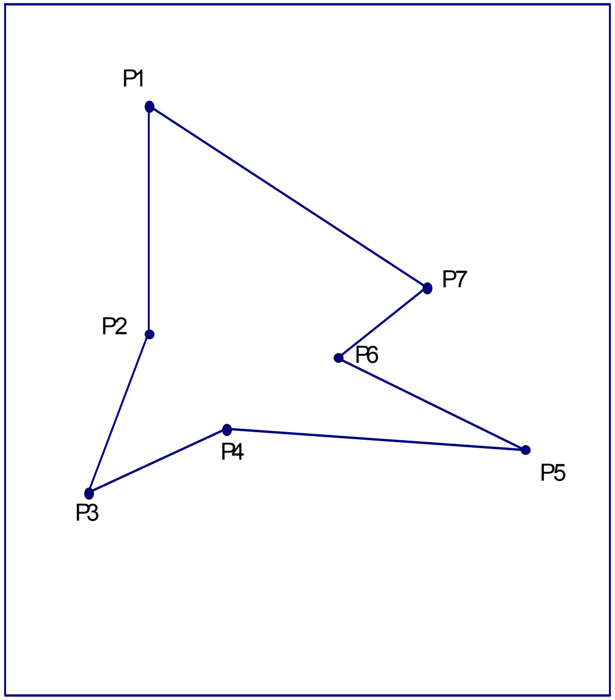
  for every three points A, B, and C of S

  there exists a point M such that all three segments MA, MB, and MC are completely contained in S.

  Then S is star-shaped.

Krasnosel'skiĭ's Theorem

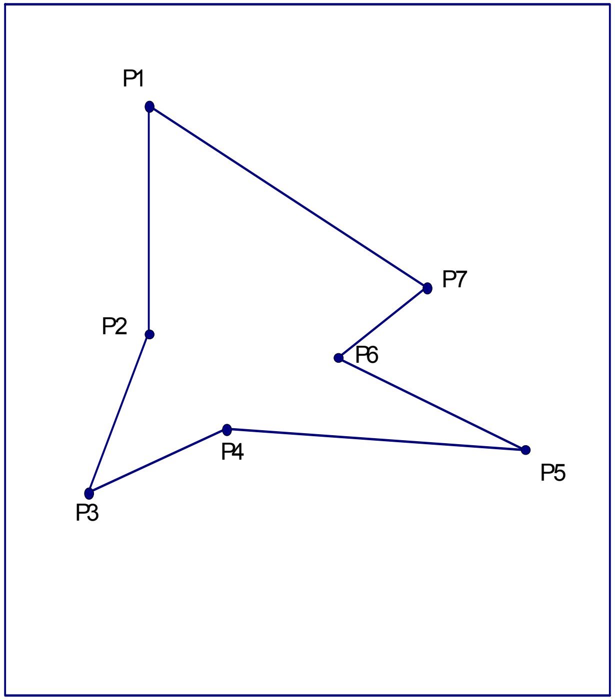
- Proof. Let P_1, P_2, … , P_n be the vertices of the polygon S in counter clockwise order.



Krasnosel'skiĭ's Theorem

- Proof. Let P_1, P_2, … , P_n be the vertices of the polygon S in counter clockwise order.

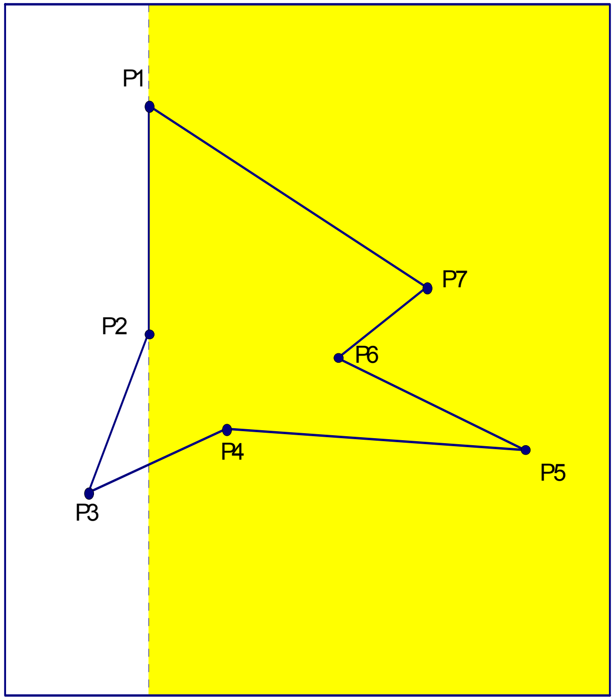
- For each side P_iP_{i+1} consider the half plane to its left.



Krasnosel'skiĭ's Theorem

- For each side P_iP_{i+1} consider the half plane to its left.

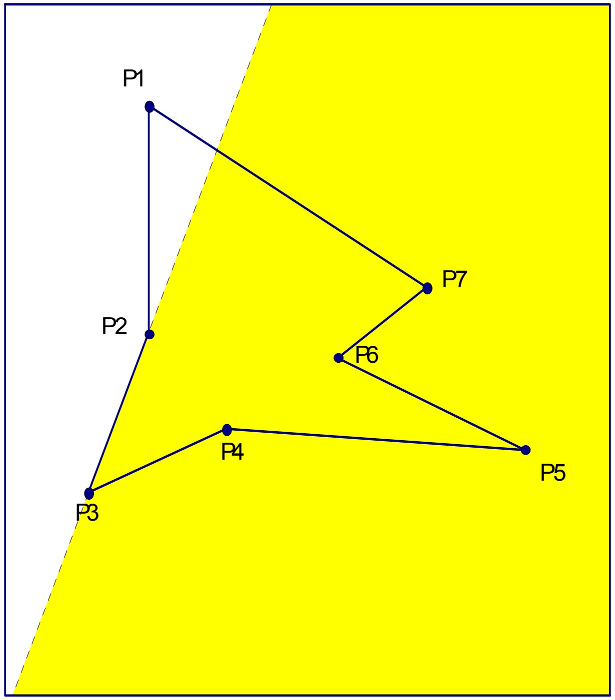
P_1P_2



Krasnosel'skiĭ's Theorem

- For each side P_iP_{i+1} consider the half plane to its left.

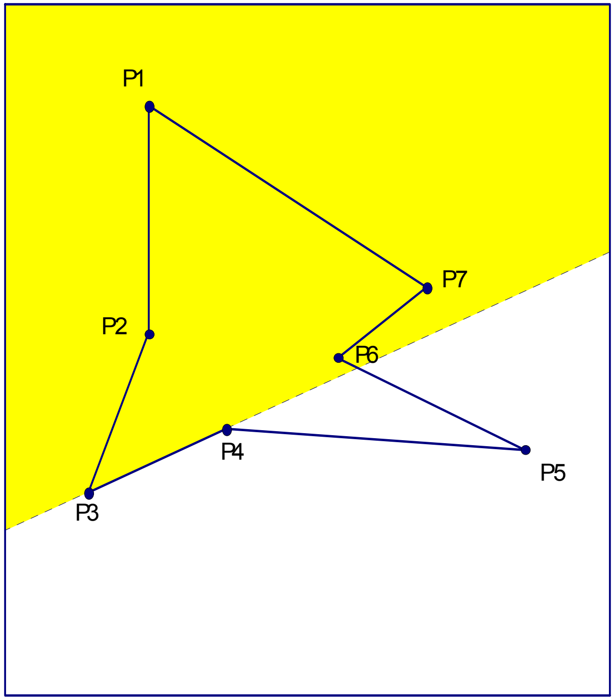
P_2P_3



Krasnosel'skiĭ's Theorem

- For each side P_iP_{i+1} consider the half plane to its left.

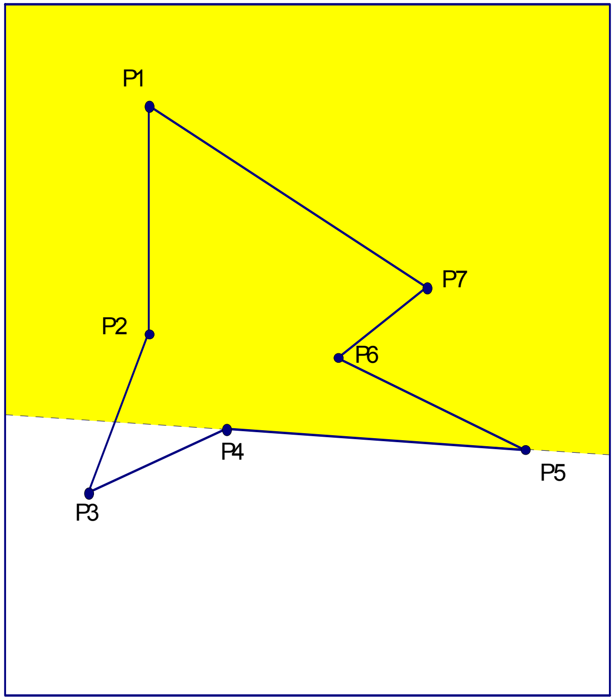
P_3P_4



Krasnosel'skiĭ's Theorem

- For each side P_iP_{i+1} consider the half plane to its left.

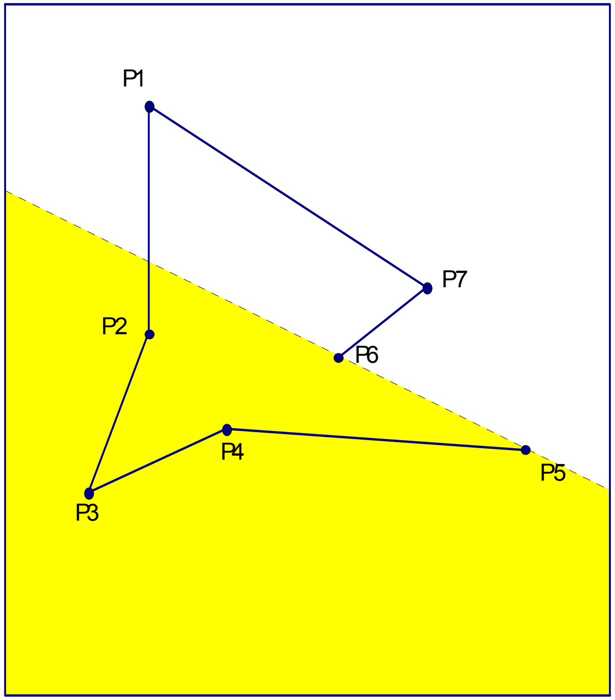
P_4P_5



Krasnosel'skiĭ's Theorem

- For each side P_iP_{i+1} consider the half plane to its left.

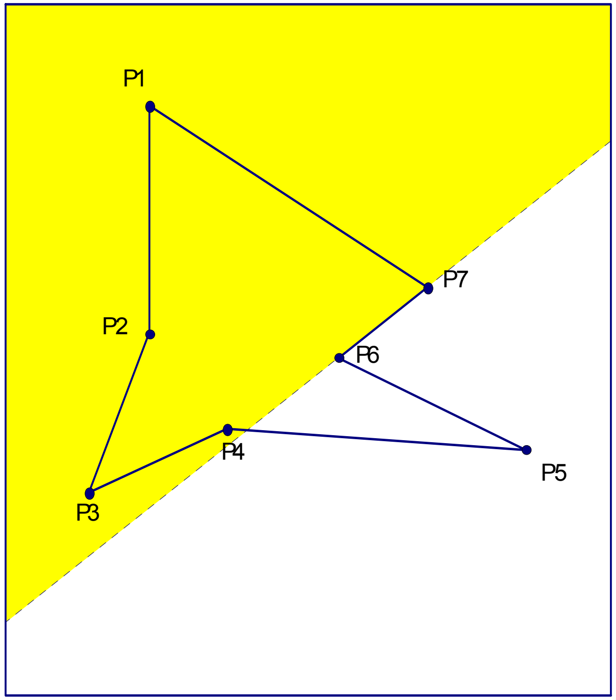
P_5P_6



Krasnosel'skiĭ's Theorem

- For each side P_iP_{i+1} consider the half plane to its left.

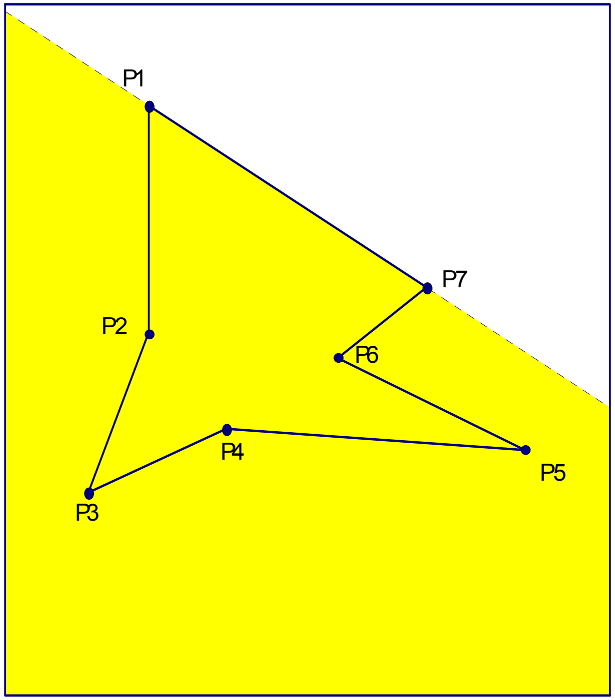
P_6P_7



Krasnosel'skiĭ's Theorem

- For each side P_iP_{i+1} consider the half plane to its left.

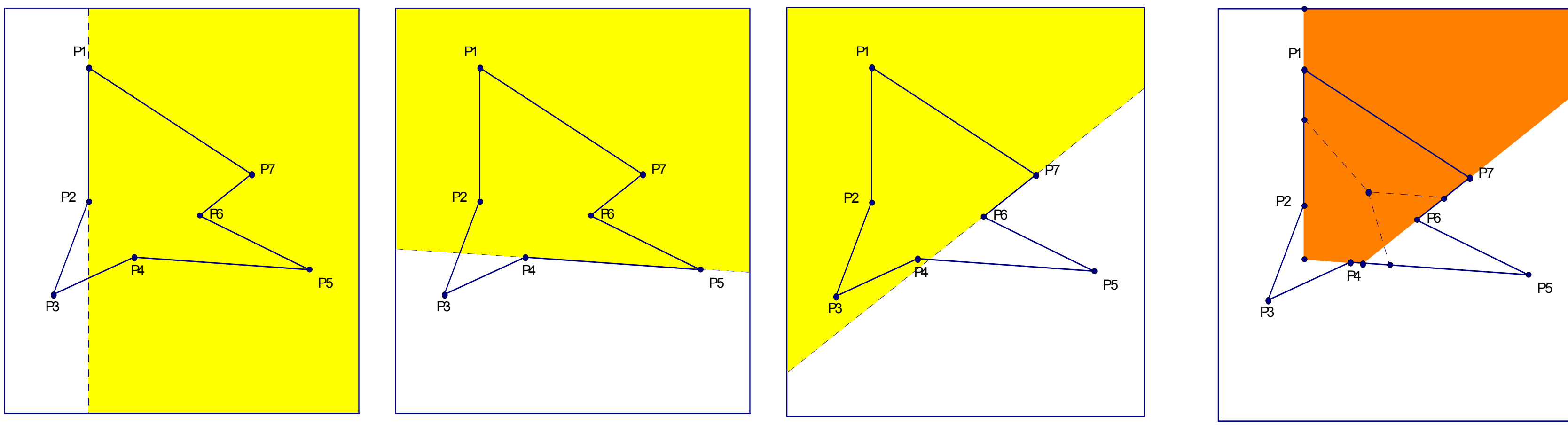
P_7P_1



Krasnosel'skiĭ's Theorem

These n half planes:

- are convex
- satisfy Helly's condition



$$P_1P_2 \quad \cap \quad P_4P_5 \quad \cap \quad P_6P_7$$

Krasnosel’skiĭ’s Theorem

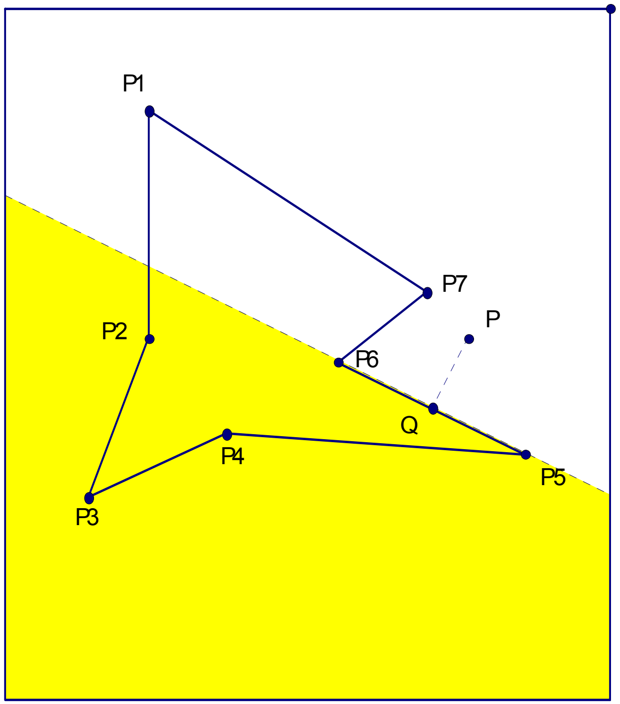
- Thus the n half planes have a point P in common.
- Assume that P is not in S then

Let Q be the point in S closest to P.

Q belongs to a side of S, say P_kP_{k+1}.

But then P doesn’t belong to the half-plane to the left of P_kP_{k+1}, getting a contradiction.

- Therefore P is in S and our proof is complete.



Guards in Art Galleries

- In 1973, Victor Klee (U. of Washington) posed the following problem:

  How many guards are required
  to guard an art gallery?

- In other words, he was interested in the number of points (guards) needed to guard a simple plane polygon (art gallery or museum).

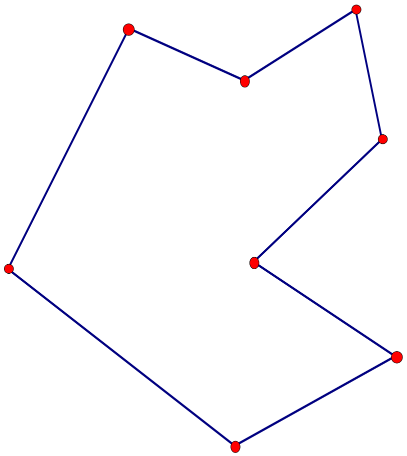
Guards in Art Galleries

- In 1975, Vašek Chvátal (Rutgers University) proved the following theorem:

Theorem. At most $\lfloor (n/3) \rfloor$ guards are necessary to guard an art gallery with n walls, (represented as a simple polygon with n sides).

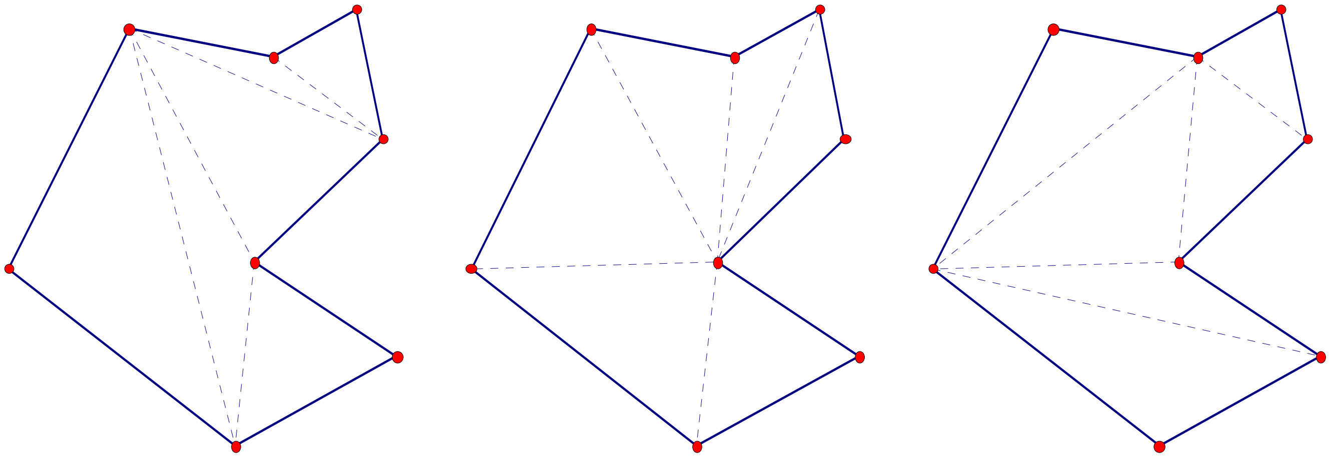
Guards in Art Galleries

- Proof. A “book proof” by Steve Fisk, 1978 (Bowdoin College). Consider any polygon P.



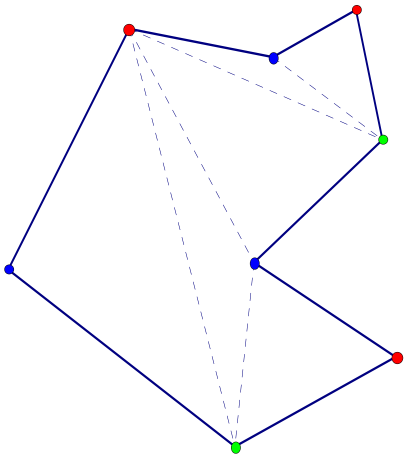
Guards in Art Galleries

- P can be triangulated in several ways.



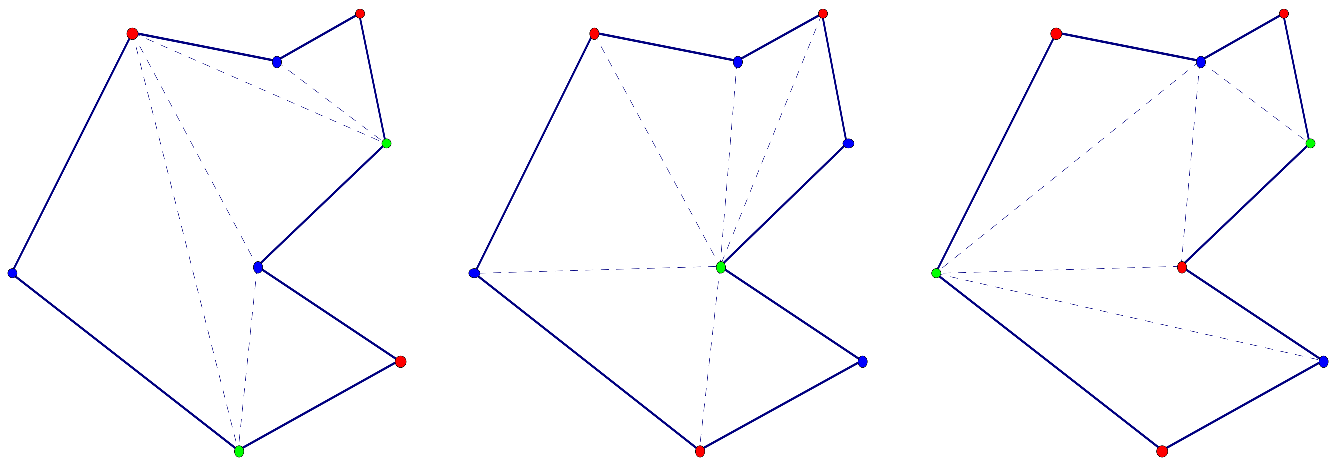
Guards in Art Galleries

- For any triangulation, the vertices of P can be colored with red, blue, and green in such a way that each triangle has a vertex of each color.



Guards in Art Galleries

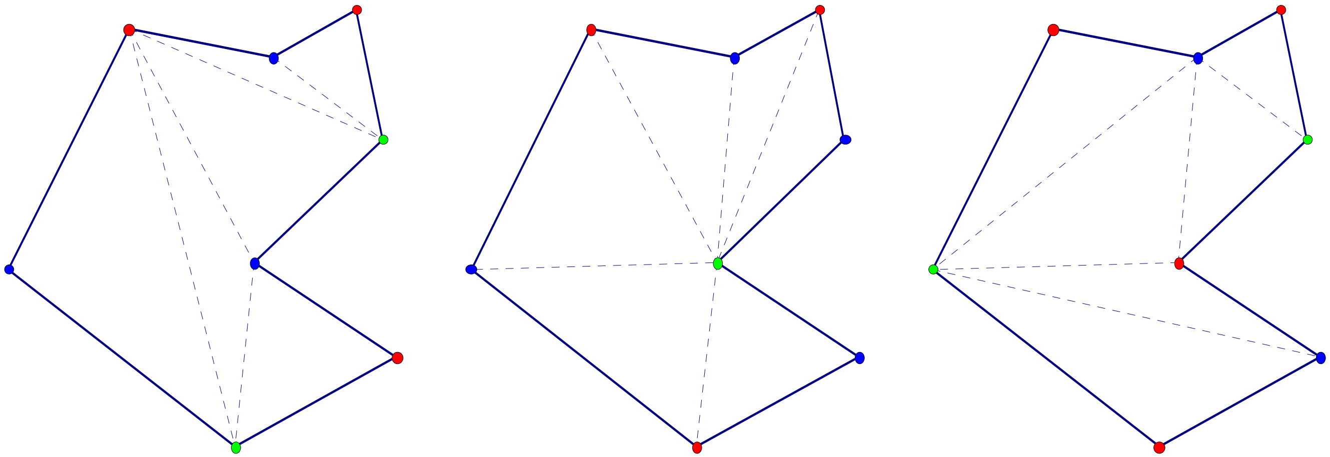
- Since there are n vertices, there is one color that is used for at most $\lfloor (n/3) \rfloor$ vertices.



Red	3	3	4
Blue	3	4	2
Green	2	1	2

Guards in Art Galleries

- Placing the guards on vertices with that color concludes our proof.



Applications

- Placement of radio antennas
- Architecture
- Urban planning
- Mobile robotics
- Ultrasonography
- Sensors

References

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