The Harmonic Series, the Natural Logarithm, and the Euler Mascheroni Constant

In this project we investigate the harmonic series

$$\sum_{k=1}^{n} \frac{1}{k}$$

This series can be graphically represented as the area under the graph of the function

$$f(x) = \frac{1}{\lfloor x \rfloor},$$

for $x \geq 1$ as shown below. We can compare this step function with the continuous function

$$g(x) = \frac{1}{x},$$

also shown in the graph.
Exercises

1. Use the graph to prove that:

\[ 1 + \ln n = 1 + \int_1^n \frac{1}{x} \, dx \geq \sum_{k=1}^n \frac{1}{k} \geq \int_1^n \frac{1}{x} \, dx = \ln n. \]

2. Use this inequality to show that the sequence

\[ a_n = \sum_{k=1}^n \frac{1}{k} - \ln n \]

is bounded.

3. Prove that \( a_n \) is increasing (look at the graph).

4. Prove that this sequence has a limit.

5. Use Maple to approximate that limit to 5 digits.

6. This limit is known as the Euler-Mascheroni Constant or Euler’s Constant. Research the history of this constant on the internet or in the library and write an essay (around 200 words) about it. Make sure to carefully list your references.