Special Functions

Preliminary Example:

Only very few functions can be expressed as simple formulas – polynomials, rational functions, roots, exponentials and trigonometric functions. A simple problem as finding the anti-derivative of $\frac{1}{x}$ can’t be solved with such simple functions. The solution is to define new functions. In this case we define a new symbol

$$\ln x = \int_1^x \frac{dt}{t}.$$  

It turns out that this new function is also the inverse of the exponential function. In this assignment, we will investigate the properties of two special functions which are defined by integrals:

Exercises: An Anti-derivative

Define the normal distribution function

$$\Phi(x) = \int_0^x e^{-\frac{t^2}{2}} \, dt.$$  

1. Prove that this function is an increasing function, which has an inflection point at $x = 0$.

2. Plot the graph of the function.
3. From the graph guess

\[ \lim_{x \to \infty} \Phi(x). \]

4. Plot the graph of the error function

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt. \]

5. Show that

\[ \text{erf}(x) = \frac{1}{\sqrt{2\pi}} \Phi(x). \]

(Hint: Compute the derivative of their difference and conclude that their difference is constant.)

6. Show that \( \text{erf}(-x) = -\text{erf}(x) \). In other words, the error function is an odd function.

**Exercises: The Gamma Function**

Consider the function defined by

\[ \Gamma(x) = \int_0^\infty t^x e^{-t} \, dt. \]

1. Show that this integral is finite for all \( x > 0 \).

2. Compute \( \Gamma(0), \Gamma(1), \Gamma(2), \Gamma(3), \Gamma(4), \Gamma(5) \)

3. Show that \( \Gamma(x) \) has a positive derivative for \( x > 0 \) and is therefore an increasing function.

4. Show that \( \Gamma(x+1) = (x+1)\Gamma(x) \) (Hint: Integration by parts).

5. Show that for positive integers we have: \( n! = \Gamma(n) \).

6. This function is known as the *Gamma function* which is very important in higher mathematics. It is built into Scientific Notebook by using the symbol \( \Gamma(x) \). Plot the graph of this function.