The Chebyshev Polynomials

To do this assignment, it will be helpful to understand assignment 9.

**Definition 1** *The Chebyshev polynomials* $T_n$ are given by

$$T_n(x) = \cos(n \arccos x),$$

for $n = 0, 1, 2, 3, \cdots$.

1. What are the domain and range of these functions?

2. We know that $T_0(x) = 1$ and $T_1(x) = x$. Express $T_2$ explicitly as a quadratic polynomial and $T_3$ as a cubic polynomial.

3. Show that

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

for $n \geq 1$.

4. Use problem 3 to show that $T_n$ is a polynomial of degree $n$.

5. Use problems 3 and 4 to express $T_4, T_5, T_6$ and $T_7$ explicitly as polynomials.

6. What are the zeros of $T_n$? At what numbers does $T_n$ have local maximum and local minimum values?

7. Graph $T_2, T_3, T_4$ and $T_5$ on a common screen.

8. Graph $T_5, T_6$ and $T_7$ on a common screen.

9. Based on your observations from problems 7 and 8, how are the zeros of $T_n$ related to the zeros of $T_{n-1}$? What about the $x$-coordinates of the maximum and minimum values?

10. Based on your graphs in problems 7 and 8, what can you say about

$$\int_{-1}^{1} T_n(x) \, dx$$

when $n$ is odd and when $n$ is even?

11. Use the substitution $u = \arccos x$ to evaluate the integral in problem 10.

12. The family of functions $f(x) = \cos(n \arccos x)$ are defined even when $c$ is not an integer (but then $f$ is not a polynomial). Describe how the graph of $f$ changes as $c$ increases.