The Mean Value Theorem

The theorem is commonly given as

**Theorem 1** Suppose that \( f \) is a continuous function on a closed interval \([a, b]\) and differentiable on the open interval \((a, b)\), then there exists at least one number \( c \in (a, b) \) with the property that

\[
\frac{f(b) - f(a)}{b - a} = f'(c).
\]

1. Consider the function \( f(x) = x + \frac{4}{x} \).
   (a) Graph \( f(x) \) in the window \([0, 10]\) by \([0, 10]\).
   (b) Graph the secant line that passes through the points \((1, 5)\) and \((8, 8.5)\) on the same screen with \( f(x) \).
   (c) Find the number \( c \) that satisfies the conclusion of the Mean Value Theorem for \( f(x) \) on the interval \([1, 8]\).
   Then graph the tangent line at the point \((c, f(c))\).
   (d) What is the relationship between the secant line and the tangent line?

2. Consider the function \( g(x) = x^3 - 2x \).
   (a) Graph \( g(x) \) in the window \([-3, 3]\) by \([-5, 5]\).
   (b) Graph the secant line that passes through the points \((-2, -4)\) and \((2, 4)\) on the same screen with \( g(x) \).
   (c) Use the graph to estimate the \( x \)-coordinates of then points there the tangent line is parallel to the secant line.
   (d) Find the numbers \( c \) that satisfy the conclusion of the Mean Value Theorem for \( g(x) \) on the interval \([-2, 2]\).
   Then graph the tangent line at the point \((c, f(c))\).
   (e) Compare your answers in parts (b) and (c).

3. Use the theorem to show that \(|\sin x - \sin y| \leq |x - y|\).

4. Use the theorem to prove that if \(|f'(x)| \leq M\) for all \( x \in (a, b) \) and if \( x_1 \) and \( x_2 \in (a, b) \), then

\[
|f(x_2) - f(x_1)| \leq M|x_2 - x_1|.
\]

5. For the functions below, tell why you cannot apply the Mean Value Theorem.
   (a) \( f(x) = |x| \) on \([-2, 2]\).
   (b) \( g(x) = \frac{1}{x-1} \) on \([0, 2]\).
   (c) \( h(x) = \csc x \) on \([-\pi, \pi]\).
   (d) \( i(x) = |x| \) on \([1, 2]\)