Local and Global Extrema

Example

An extremum of a function is either its maximum or its minimum. A point \( x_0 \) is called a relative maximum of the function \( f \) if \( f(x_0) \geq f(x) \) for all values of \( x \) close to \( x_0 \). In other words, the value of the function at \( x_0 \) is larger than at neighboring points. At a local minimum the value of the function is smaller than at neighboring points. Let \( f(x) = \sin x^2 \) on the interval \([0, \sqrt{2\pi}]\). The graph of this function is shown below.

![Graph of f(x) = sin x^2 on [0, \sqrt{2\pi}]](image)

Figure 1: The graph of \( f(x) = \sin x^2 \) on \([0, \sqrt{2\pi}]\).

This function has two local minima and two local maxima. The candidates for these local extrema are the end points of the interval and the points where the graph has a horizontal tangent line. Since \([0, \sqrt{2\pi}]\) is a closed interval, the end points are included. To
find the local extrema we first find the critical points, i.e., the points where the derivative vanishes or does not exist. To do this we solve

\[ f'(x) = 2 \cos^2 x = 0, \]

The solutions are:

\[ \left\{ x = \frac{1}{2} \sqrt{2} \sqrt{\pi} \right\}, \left\{ x = -\frac{1}{2} \sqrt{2} \sqrt{\pi} \right\}, \{ x = 0 \}. \]

The computer returns three solutions, only two of which are in the interval \([0, \sqrt{2\pi}]\). However, it does not find the third solution in the interval at

\[ x = \sqrt{\frac{3\pi}{2}}. \]

From this and the graph, we observe that \( f \) has local minima at \( x = 0 \), since \( f(0) = 0 \) and at

\[ x = \sqrt{\frac{3\pi}{2}} \quad \text{since} \quad f \left( \sqrt{\frac{3\pi}{2}} \right) = -1. \]

We have local maxima at

\[ x = \frac{1}{2} \sqrt{2} \sqrt{\pi} \quad \text{since} \quad f \left( \frac{1}{2} \sqrt{2} \sqrt{\pi} \right) = 1 \]

and at \( x = \sqrt{2\pi} \) since \( f \left( \sqrt{2\pi} \right) = 0. \)

A function has a global maximum (or minimum) at \( x_0 \) in an interval \([a, b]\), if \( f(x_0) \geq f(x) \) (or \( f(x_0) \leq f(x) \)) for all \( a \leq x \leq b \). A continuous function in a closed interval has always a global maximum and a global minimum.

In this case (the closed interval) the global maximum is the largest local maximum, the global minimum is the smallest local minimum. In our example the largest local maximum is at

\[ x = \frac{1}{2} \sqrt{2} \sqrt{\pi}, \]

since

\[ f \left( \frac{1}{2} \sqrt{2} \sqrt{\pi} \right) = 1 > 0 = f \left( \sqrt{2\pi} \right). \]
The smallest local minimum is at
\[ x = \sqrt{\frac{3\pi}{2}}, \]
where
\[ f \left( \sqrt{\frac{3\pi}{2}} \right) = -1. \]

If our interval of interest would have been \([0, \sqrt{\pi})\), then the function would have a local and global minimum, but neither a local nor a global maximum.

**Exercises**

In the following exercises find all global and local extrema of the given function in the given interval. Explain your results carefully.

1. \( g(x) = x^2 (1 - x^2) \) for \(-2 \leq x < 3\).
2. \( g(x) = x^2 (1 - x^2) \) for \(-2 < x < 3\).
3. \( g(x) = x^2 (1 - x^2) \) for \(-2 \leq x \leq 3\).
4. \( h(x) = \ln(x^2 + 1) \) for \(-2 \leq x < 3\).
5. \( k(x) = \frac{x - 1}{x + 1} \) for \(-2 \leq x \leq 3\).
6. \( l(x) = x + \frac{1}{x} \) for \(0 < x < 2\).
7. \( l(x) = x + \frac{1}{x} \) for \(0 < x \leq 2\).