Trigonometric Polynomials

Define the trigonometric polynomial as a finite linear combination of \( \sin(nx) \) and \( \cos(nx) \) with \( n \) a natural number. It has been established that

\[
s = f(t) = 0.78540 - 0.63662 \cos 2t - 0.07074 \cos 6t - 0.02546 \cos 10t - 0.01299 \cos 14t
\]
gives an approximation of the sawtooth function \( s = g(t) \) on the interval \([-\pi, \pi]\). How well does the derivative of \( f \) approximate the derivative of \( g \) at the points where \( dg/dt \) is defined? To find out, carry out the following steps.

\[
\begin{align*}
\text{Figure 1: } f(t) \text{ is in black and } g(t) \text{ is in blue.}
\end{align*}
\]

Exercises

1. Graph \( dg/dt \) (where defined) over \([-\pi, \pi]\).

2. Find \( df/dt \).

3. Graph \( df/dt \). Where does the approximation of \( dg/dt \) by \( df/dt \) seem to be the best? Least good?

Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them as we shall see.

So we notice that the trigonometric polynomial \( f(t) \) that approximated the sawtooth function \( g(t) \) on \([-\pi, \pi]\) has a derivative that approximated the sawtooth function. It is possible, however,
for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the trigonometric polynomial

\[ s = h(t) = 1.2732 \sin 2t + 0.4244 \sin 6t + 0.25465 \sin 10t + 0.18189 \sin 14t + 0.14147 \sin 18t \]

approximates the step function \( s = k(t) \).

Figure 2: \( h(t) \) is in black and \( k(t) \) is in blue.

Notice that the derivatives of \( h \) look nothing like the derivatives of \( k \).

Exercises

1. Graph \( \frac{dk}{dt} \) (where defined) over \([-\pi, \pi]\).

2. Find \( \frac{dh}{dt} \).

3. Graph \( \frac{dh}{dt} \) to see how badly the graph fits the graph of \( \frac{dk}{dt} \). Comment on what you see.