Linearization Approximations

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with. The approximation of functions discussed here are called linearizations.

**Definition 1** If $f$ is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the linearization of $f$ at $a$.

**Example 2** Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$.

**Solution.** Since

$$f'(x) = \frac{1}{2}(1 + x)^{-\frac{1}{2}},$$

we have $f(0) = 1$, $f'(0) = \frac{1}{2}$, and

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

Notice how accurate our approximation is in the table below.

| Approximation | $|$True Value – Approximation$|$
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<tr>
<td>$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$</td>
<td>$&lt; 10^{-2}$</td>
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<tr>
<td>$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$</td>
<td>$&lt; 10^{-3}$</td>
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<tr>
<td>$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$</td>
<td>$&lt; 10^{-5}$</td>
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Exercises

In the problems below, estimate the magnitude of the error in using the linearization in place of the function over a specified interval \( I \). Perform the following steps.

- Plot the function \( f \) over \( I \).
- Find the linearization \( L \) of the function at the point \( a \).
- Plot \( f \) and \( L \) together on a single graph.
- Plot the absolute error \( |f(x) - L(x)| \) over \( I \) and find its maximum value.
- From your graph, the the above step, estimate as large a \( \delta > 0 \) as you can satisfying

\[
|x - a| < \delta \quad \text{implies} \quad |f(x) - L(x)| < \varepsilon
\]

for \( \varepsilon = 0.5, 0.1, \) and 0.01. Then check graphically to see your \( \delta \)-estimate holds true.

1. \( f(x) = x^3 + x^2 - 2x, \) \([-1, 2], a = 1.\)
2. \( f(x) = \frac{x - 1}{4x^2 + 1}, \) \([-\frac{3}{4}, 1]\), \( a = \frac{1}{2}.\)
3. \( f(x) = x^{\frac{3}{2}}(x - 2), \) \([-2, 3], a = 2.\)
4. \( f(x) = \sqrt{x} - \sin x, \) \([0, 2\pi], a = 2.\)