MATH 350 EXAM #3 Fall 2004

(SAMPLE)

1. Consider the following sequence

$$u_n \;=\; \sum_{k=0}^{n} \frac{1}{n+k} \;=\; \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} \;; \quad \forall n \in \mathbf{N}$$

   a. Show that (u_n) is a convergent sequence. We set $l \;=\; \lim_{n\to\infty} u_n$.

   b. Consider $f : [0\;,\;1] \longrightarrow \mathbf{R}$, such that $f(0) = 0$ and f differentiable at 0^+. We set $\alpha \;=\; f^{'}(0^+)$ and we introduce the sequence (θ_n) defined by

$$\theta_n \;=\; \sum_{k=0}^{n} f(\frac{1}{n+k}) \;=\; f(\frac{1}{n}) + f(\frac{1}{n+1}) + \cdots + f(\frac{1}{2n-1}) + f(\frac{1}{2n}) \;; \quad \forall n \in \mathbf{N}$$

   Prove that

$$\lim_{n\to\infty} \theta_n \;=\; \alpha l$$

   c. Use part (b) with $f(x) = \log(1+x)$ to deduce the value of the limit l.

2. Consider $f, g : [a\;,\;b] \longrightarrow \mathbf{R}$. Assume that f and g are continuous on $[a\;,\;b]$ and differentiable on $(a\;,\;b)$.

   a. For $x \in [a\;,\;b]$, consider the following function

$$\Phi(x) \;=\; \lambda(f(x) - f(a)) + \mu(g(x) - g(a))$$

   Find λ and μ such that $\Phi(a) = \Phi(b)$.

   b. Deduce the existence of $c \in (a\;,\;b)$ such that

$$(f(b) - f(a))g^{'}(c) \;=\; (g(b) - g(a))f^{'}(c)$$

3. Apply the mean value theorem to the function $\log|\log|x||$ on a suitable interval to find the following limit

$$\lim_{n\to\infty} \left(\frac{1}{2\log 2} + \frac{1}{3\log 3} + \cdots + \frac{1}{(n-1)\log(n-1)} + \frac{1}{n\log n} \right)$$

4. For a real number a and $h > 0$, consider $f : [a-h\ ,\ a+h] \longrightarrow \mathbf{R}$. Assume that f is continuous on $[a-h\ ,\ a+h]$ and differentiable on $(a-h\ ,\ a+h)$. Prove that

a. $\exists\ \mu \in (0\ ,\ 1) : \quad f(a+h) - f(a-h) \ = \ h\left[f^{'}(a+\mu h) + f^{'}(a-\mu h)\right]$

b. $\exists\ \theta \in (0\ ,\ 1) : \quad f(a+h) - 2f(a) + f(a-h) \ = \ h\left[f^{'}(a+\theta h) - f^{'}(a-\theta h)\right]$

5. Consider f a $\mathcal{C}^1$ function on interval containing a.

a. For $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}$, and $p \in \mathbf{N}$, find the following limit

$$\lim_{h \to 0} \frac{f^p(a+\alpha h) - f^p(a+\beta h)}{h}$$

b. Find conditions on f so that the limit in part(a) exists for $p \in \mathbf{R}$.

6. a. Show that

$$1 + x \leq e^x \ ; \quad \forall x \in \mathbf{R}$$

b. For $k \in [0\ ,\ 1)$, consider the following sequence

$$u_n \ = \ (1+k)(1+k^2)\cdots(1+k^n)\ ; \quad \forall n \in \mathbf{N}$$

Prove that (u_n) is a convergent sequence.

7. Consider $f : [a\ ,\ b] \longrightarrow \mathbf{R}$. Assume that f is a $\mathcal{C}^n$ function on $[a\ ,\ b]$, and $\{a_0, a_1, \cdot\cdot, a_{n-1}, a_n\}$ a set of $n+1$ real numbers such that $a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b$ and

$$f(a_0) \ = \ f(a_1) \ = \ f(a_2) \ = \ \cdots \ = \ f(a_{n-1}) \ = \ f(a_n) \ = \ 0$$

Show that $\forall t \in [a\ ,\ b]$, $\exists\ c \in (a\ ,\ b)$ such that

$$f(t) \ = \ (t-a_0)(t-a_1)(t-a_2)\cdots(t-a_{n-1})(t-a_n)\frac{f^{(n+1)}(c)}{(n+1)!}$$

8. Find all $\mathcal{C}^2$ functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$\exists\ \theta \in [0\ ,\ 1]\ ,\ \forall x \in \mathbf{R} \text{ and } \forall h \in \mathbf{R}\ : \quad f(x+h) \ = \ f(x) + hf^{'}(x+\theta h)$$

9. Evaluate the following limits.

a. $\displaystyle\lim_{x \to \frac{\pi}{2}} (\pi - 2x)\tan x$

b. $\displaystyle\lim_{x \to a} \frac{\sin^2 x - \sin^2 a}{x^2 - a^2}$

c. $\displaystyle\lim_{x \to \infty} x^2\left(e^{\frac{1}{x}} - e^{\frac{1}{x+1}}\right)$

d. $\displaystyle\lim_{x \to 0} \frac{\sin \frac{1}{x}}{e^{\frac{1}{x}} + 1}$

e. $\displaystyle\lim_{x \to \infty} \sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}$