MATH 350 EXAM #2 Fall 2004

(SAMPLE)

1. Decide which of the following limits exist and which do not. Prove that your answer is correct.

   a. $\lim_{x\to 0} \cos\frac{1}{x}$

   b. $\lim_{x\to 0} \frac{1}{\log x}$

   c. $\lim_{x\to 0} x^n \sin\frac{1}{x^2}; \quad n \geq 0$

2. Evaluate the following limits when they exist.

   a. $\lim_{x\to \frac{\pi}{2}^-} \frac{\tan x}{x}$

   b. $\lim_{x\to 0^+} \frac{\sqrt{1-\cos x}}{\sin x}$

   b. $\lim_{x\to +\infty} \arctan x$

   b. $\lim_{x\to 0^+} \frac{\sqrt{1-\cos x}}{\sin x}$

   b. $\lim_{x\to -\infty} x^2 \sin x$

   c. $\lim_{x\to 0} \frac{\sin x \log(1+x^2)}{x\tan x}$

   (You may assume that $\lim_{u\to 0} \frac{\log(1+u)}{u} = 1$

3. State whether each of the following statements are **TRUE** or **FALSE**. You do need to show your work when your answer is **FALSE** only.

   a. If (x_n) is Cauchy, then there is *at most* a subsequence that is Cauchy.

   b. Let I be an interval and (x_n) is Cauchy in I. If $f : I \longrightarrow \mathbf{R}$ is continuous, then $(f(x_n))$ is *necessarily* Cauchy

   c. Let I be an interval and (x_n) be a convergent sequence in I. If $f : I \longrightarrow \mathbf{R}$ is continuous, then $(f(x_n))$ is *necessarily* converges

   d. Let I be an interval and $f : I \longrightarrow \mathbf{R}$. Assume that there is a sequence (x_n) in I that converges to $l \in I$. If $\lim_{n\to\infty} f(x_n) = f(l)$, then f is *necessarily* continuous.

   e. A continuous function f on a bounded interval I is *necessarily* bounded.

f. $f : [0\ ,\ 1] \longrightarrow [0\ ,\ 1]$. Then, there is *necessarily* $x_0 \in [0\ ,\ 1]$ such that $f(x_0) = x_0$.

g. A polynomial of degree n ($n \geq 0$) is uniformly continuous on $\mathbf{R}$.

4. Suppose that $f : [a\ ,\ b] \longrightarrow \mathbf{R}$ is continuous and $f(a) \neq f(b)$. Prove that if p and q are two positive real numbers, then

$$\exists\ c \in]a\ ,\ b[, \quad pf(a) + qf(b) \ = \ (p+q)f(c)$$

5. Suppose that $f : [a\ ,\ b] \longrightarrow [a\ ,\ b]$ is continuous. Prove that

$$\exists\ c \in [a\ ,\ b], \quad f(c) \ = \ c$$

Note that c is called a fixed point.

6. Let E be a nonempty subset of $\mathbf{R}$ and $f : E \longrightarrow \mathbf{R}$. f is said to be Lipschitz function if

$$\exists\ k \in \mathbf{R} \text{ such that } \forall x \in E \text{ and } \forall y \in E : |f(x) - f(y)| \leq k\,|x - y|$$

Prove that a Lipschitz function is uniformly continuous.

7. Suppose that $x_n \in \mathbf{N}$ for $n \in \mathbf{N}$. If (x_n) is Cauchy, prove that there are numbers a and N such that

$$x_n \ = \ a\ ; \quad \forall n \geq N$$

8. Suppose that $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous and satisfies

$$\forall x, y \in \mathbf{Q}\ , \quad f(x+y) \ = \ f(x) + f(y)$$

Prove that

$$\exists\ a \in \mathbf{R} \text{ such that } \forall x \in \mathbf{R} \quad f(x) \ = \ ax$$

9. Suppose that $f : [a\ ,\ \infty) \rightarrow \mathbf{R}$ is continuous and there is l such that $\lim\limits_{x \to +\infty} f(x) = l$. Prove that f is bounded on $[a\ ,\ \infty)$.

10. Let E be a nonempty subset of $\mathbf{R}$ and $f : E \rightarrow \mathbf{R}$ is uniformly continuous. Assume (x_n) is Cauchy. Prove that $(f(x_n))$ is Cauchy.

11. Let I be a bounded interval and $f : I \longrightarrow \mathbf{R}$. Prove that if f is uniformly continuous on I, then f is bounded on I. What happens if I is unbounded?

12. Suppose that $f : \mathbf{R} \rightarrow \mathbf{R}$ is continuous and there is l_1 and l_2 such that $\lim\limits_{x \to +\infty} f(x) = l_1$ and $\lim\limits_{x \to -\infty} f(x) = l_2$. Prove that f is uniformly continuous on $\mathbf{R}$.