MATH 262 | EXAM #1 | Spring 2006

(SAMPLE)

1. **Section 1.1**

a. Find all the solutions of the following linear system

$$\begin{cases} x + 2y + 3z &= 1 \\ 3x + 2y + z &= 1 \\ 7x + 2y - 3z &= 1 \end{cases}$$

b. Find all the solutions of the following linear system

$$\begin{cases} x + 2y + 3z &= a \\ x + 3y + 8z &= b \\ x + 2y + 2z &= c \end{cases}$$

where a, b, and c are arbitrary constants.

c. Find all the solutions of the following linear system

$$\begin{cases} 7x - y &= \lambda x \\ -6x + 8y &= \lambda y \end{cases}$$

for *i.* $\lambda = 5$, *ii.* $\lambda = 10$, and *iii.* $\lambda = 15$.

d. Consider the linear system

$$\begin{cases} x + y - z &= -2 \\ 3x - 5y + 13z &= 18 \\ x - 2y + 5z &= k \end{cases}$$

where k is an arbitrary constant.

i. For which value(s) of k, does this system have one or infinitely many solutions?
ii. For each value of k you found in part (i), how many solutions does this system have?
iii. Find all the solutions for each value of k.

2. **Section 1.2**

a. Solve the following system, using Gauss-Jordan elimination, for the variables x_1, x_2, x_3, x_4, and x_5.

$$\begin{cases} x_2 + 2x_4 + 3x_5 & = & 0 \\ \\ 4x_4 + 8x_5 & = & 0 \end{cases}$$

b. Find the polynomial of degree 3 [a polynomial of the form $f(t) = a + bt + ct^2 + dt^3$] whose graph goes through the points $(0,1), (1,0), (-1,0)$, and $(2,-15)$. Sketch the graph of this cubic.

c. Find the polynomial $f(t)$ of degree 3 such that $f(1) = 1$, $f(2) = 5$, $f'(1) = 2$, and $f'(2) = 9$, where $f'(t)$ is the derivative of $f(t)$. Graph this polynomial.

d. Find all vectors in $\mathbf{R}^4$ that are perpendicular to the three vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ 9 \\ 7 \end{bmatrix}$$

3. **Section 1.3**

a. Find the rank of the matrix

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

b. Compute the following dot product (if defined)

$$\begin{bmatrix} 1 & 9 & 9 & 7 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$$

c. Compute the product $A\vec{x}$ two ways: in term of the columns of A and in terms of the rows of A

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

d. Compute the product $A\vec{x}$ (if defined)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

e. If the rank of a 5×3 matrix A is 3, what is $rref(A)$?

4. **Section 2.1**

a. Find the matrix of the linear transformation

$$\begin{cases} y_1 &= 9x_1 + 3x_2 - 3x_3 \\ y_2 &= 2x_1 - 9x_2 + x_3 \\ y_3 &= 4x_1 - 9x_2 - 2x_3 \\ y_4 &= 5x_1 + x_2 + 5x_3 \end{cases}$$

b. Consider the transformation T from $\mathbf{R}^2$ to $\mathbf{R}^3$ given by

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Is this transformation linear? If so, find its matrix.

c. i. For which values of the constant k is the matrix $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$ invertible?

ii. For which values of the constant k are all entries of $\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$ integers?

d. For which values of the constants a and b is the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ invertible? What is the inverse in this case?

5. **Section 2.2**

a. Let L be the line in $\mathbf{R}^3$ that consists of all scalar multiples of $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

b. Let L be the line in $\mathbf{R}^3$ that consists of all scalar multiples of $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ about the L.

c. Suppose a line L in $\mathbf{R}^2$ contains the nonzero vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Find the matrix A of the linear transformation $T(\vec{x}) = proj_L\vec{x}$. Gives the entries of A in terms of v_1 and v_2.

d. Let T and L be transformations from $\mathbf{R}^n$ to $\mathbf{R}^n$. Suppose L is the inverse of T; that is

$$T(L(\vec{x})) = \vec{x} \quad \text{and} \quad L(T(\vec{x})) = \vec{x}$$

for all $\vec{x}$ in $\mathbf{R}^n$. If T is a linear transformation, is L linear as well? Hint: $\vec{x}+\vec{y} = T(L(\vec{x})) + T(L(\vec{y}))$, because T is linear. Now apply L on both sides.

e. Find a nonzero 2×2 matrix A such that $A\vec{x}$ is parallel to the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, for all $\vec{x}$ in $\mathbf{R}^2$.

6. <u>**Section 2.3**</u>

a. Decide whether the following two matrices are invertible. If they are find the inverse.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

b. Decide whether the following the following linear transformation is invertible. Find the inverse transformation if it exists

$$\begin{cases} y_1 = x_1 + 3x_2 + 3x_3 \\ y_2 = x_1 + 4x_2 + 8x_3 \\ y_3 = 2x_1 + 7x_2 + 12x_3 \end{cases}$$

c. For which values of the constant k is the following matrix invertible?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$$

d. For which choices of the constants b and c is the following matrix invertible?

$$A = \begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

e. For which choices of the constants a, b and c is the following matrix invertible?

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

f. Consider the diagonal matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

i. For which values of a, b and c is A invertible? If it is invertible, what is A^{-1}?
ii. For which values of the diagonal elements is a diagonal matrix (of arbitrary size) invertible?

7. <u>**Section 2.4**</u>

a. If possible compute the following matrix products.

i. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

ii. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

iii. $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

iv. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

v. $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

b. For two invertible $n \times n$ matrices A and B, determine which of the following formulas are necessarily true.

i. $(A - B)(A + B) \;=\; A^2 - B^2$
ii. $(ABA^{-1})^3 \;=\; AB^3A^{-1}$
iii. $ABA^{-1} \;=\; B$
iv. $A^{-1}B$ is invertible, and $(A^{-1}B)^{-1} \;=\; B^{-1}A$
v. $(A + B)^2 \;=\; A^2 + 2AB + B^2$
v. $A + B$ is invertible, and $(A + B)^{-1} \;=\; A^{-1} + B^{-1}$