MATH 350 EXAM #1 Fall 2004

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit (no textbooks, classnotes, crib sheets, or calculators).

1. (20 points) Find the limit of each of the following sequences. If a limit does not exist, write "**DNE**" for your answer.

   a. $x_n = \dfrac{1}{\sqrt{n} + (-1)^n}$

   b. $y_n = \sqrt{n+1} - \sqrt{n}$

   c. $z_n = \dfrac{1}{n} + \sin(2n)$

   d. $t_n = \dfrac{\sqrt{2n^2 - 1}}{n+1}$

   e. $u_n = \dfrac{n}{n^2+1} + \dfrac{n}{n^2+2} + \dfrac{n}{n^2+3} + \cdots + \dfrac{n}{n^2+n}$

2. (20 points) Consider the sequence (x_n) such that $\forall n \in \mathbf{N}, x_n = (-1)^n$.

   a. Using the definition, prove that (x_n) is a divergent sequence.

   b. Use an other method to establish the divergence of the sequence (x_n).

3. (20 points) State whether each of the following statements are **TRUE** or **FALSE**. You do need to show your work when your answer is **FALSE** only.

   a. A subsequence of a bounded sequence is not necessarily convergent.

   b. Let (x_n) be a divergent sequence. Then, $\lim_{n \to +\infty} x_n = +\infty$

   c. A convergent sequence is *bounded*.

   d. A bounded sequence has *at least* one convergent subsequence.

   e. If $\{I_n\}_{n \in \mathbf{N}}$ is a nested sequence of nonempty closed intervals, then $E = \bigcap_{n \in \mathbf{N}} I_n$ contains at least one number.

HEY, THERE'S MORE—TURN THE PAGE OVER!

4. (20 points) Let a be a positive number and consider the sequence (x_n) defined by

$$\begin{cases} x_1 > 0 \\ x_{n+1} = \frac{1}{2}\left(x_n + \frac{a^2}{x_n}\right) \end{cases}$$

a. Show that $x_n \geq a$; $\forall n \geq 2$

b. Show that (x_n) is a decreasing sequence.

c. Deduce that (x_n) is a convergent sequence and find its limit.

5. (20 points) Consider a and b two real numbers such that $0 < a < b$. Let (x_n) and (y_n) be two sequences defined by

$$x_1 = a \quad \text{and} \quad y_1 = b$$

and for $n \geq 1$

$$\begin{cases} x_{n+1} = \sqrt{x_n y_n} \\ y_{n+1} = \dfrac{x_n + y_n}{2} \end{cases}$$

a. Prove that $0 < x_n < y_n$; $\forall n \in \mathbf{N}$

b. Show that (x_n) is an increasing sequence.

c. Show that (y_n) is an decreasing sequence.

d. Show that (x_n) and (y_n) converge to the same limit.

6. **Bonus Questions**(10 points) Let (x_n) and (y_n) be two sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Consider the sequence (z_n) defined by

$$z_n = \sup(x_n, y_n) \; ; \quad \forall n \in \mathbf{N}$$

Prove that (z_n) is a convergent sequence and find its limit.